Uniform Laws of Large Numbers

For many estimation procedures, the first step in a proof of asymptotic normality is an argument to establish consistency. For estimators defined by some sort of maximization or minimization of a partial-sum process, consistency often follows by a simple continuity argument from an appropriate uniform law of large numbers. The maximal inequalities from Section 7 offer a painless means for establishing such uniformity results. This section will present both a uniform weak law of large numbers (convergence in probability) and a uniform strong law of large numbers (convergence almost surely).

The proof of the weak law will depend upon the following consequence of the first two lemmas from Section 3: for every finite subset $F$ of $\mathbb{R}^n$,

$$\mathbb{P}_\sigma \max_{f} |\sigma \cdot f| \leq C \max_{f} |f|_2 \sqrt{2 + \log(|F|)}.$$

(8.1)

Here $#F$ denotes the number of vectors in $F$, as usual, and $C$ is a constant derived from the inequality between $L^1$ and $L^\Psi$ norms.

(8.2) THEOREM. Let $f_1(\omega, t), f_2(\omega, t), \ldots$ be independent processes with integrable envelopes $F_1(\omega), F_2(\omega), \ldots$. If for each $\epsilon > 0$

(i) there is a finite $K$ such that

$$\frac{1}{n} \sum_{i \leq n} \mathbb{P}_{F_i} \{ F_i > K \} < \epsilon \quad \text{for all } n,$$

(ii) $\log D_1(\epsilon |F_n|, F_{n\omega}) = o_p(n),$

then

$$\frac{1}{n} \sup_{t} |S_n(\omega, t) - M_n(t)| \to 0 \quad \text{in probability}.$$

PROOF. Let us establish convergence in $L^1$. Given $\epsilon > 0$, choose $K$ as in assumption (i) and then define $f_i^*(\omega, t) = f_i(\omega, t) \{ F_i(\omega) \leq K \}$. The variables