SECTION 8

Uniform Laws of Large Numbers

For many estimation procedures, the first step in a proof of asymptotic normality is an argument to establish consistency. For estimators defined by some sort of maximization or minimization of a partial-sum process, consistency often follows by a simple continuity argument from an appropriate uniform law of large numbers. The maximal inequalities from Section 7 offer a painless means for establishing such uniformity results. This section will present both a uniform weak law of large numbers (convergence in probability) and a uniform strong law of large numbers (convergence almost surely).

The proof of the weak law will depend upon the following consequence of the first two lemmas from Section 3: *for every finite subset $\mathcal{F}$ of $\mathbb{R}^n$,*

$$\mathbb{P}_\sigma \max_{\mathcal{F}} |\boldsymbol{\sigma} \cdot \mathbf{f}| \leq C \max_{\mathcal{F}} |\mathbf{f}|_2 \sqrt{2 + \log(\#\mathcal{F})}. \tag{8.1}$$

Here $\#\mathcal{F}$ denotes the number of vectors in $\mathcal{F}$, as usual, and C is a constant derived from the inequality between $\mathcal{L}^1$ and $\mathcal{L}^\Psi$ norms.

(8.2) THEOREM. *Let $f_1(\omega, t)$, $f_2(\omega, t)$, ... be independent processes with integrable envelopes $F_1(\omega)$, $F_2(\omega)$, ... If for each $\epsilon > 0$*

(i) there is a finite K such that

$$\frac{1}{n} \sum_{i \leq n} \mathbb{P} F_i \{F_i > K\} < \epsilon \qquad \textit{for all } n,$$

(ii) $\log D_1(\epsilon |\mathbf{F}_n|, \mathcal{F}_{n\omega}) = o_p(n)$,

then

$$\frac{1}{n} \sup_t |S_n(\omega, t) - M_n(t)| \to 0 \qquad \textit{in probability.}$$

PROOF. Let us establish convergence in $\mathcal{L}^1$. Given $\epsilon > 0$, choose K as in assumption (i) and then define $f_i^*(\omega, t) = f_i(\omega, t)\{F_i(\omega) \leq K\}$. The variables