SECTION 7

Maximal Inequalities

Let us now pull together the ideas from previous sections to establish a few useful maximal inequalities for the partial-sum process S_n. To begin with, let us consider an infinite sequence of independent processes $\{f_i(\omega, t)\}$, in order to see how the bounds depend on n. This will lead us to the useful concept of a manageable triangular array of processes.

The symmetrization bound from Section 2 was stated in terms of a general convex, increasing function Φ on $\mathbb{R}^+$. The chaining inequality of Section 3 was in terms of the specific convex function given by $\Psi(x) = {}^1\!/_5 \exp(x^2)$.

Section 2 related the maximum deviation of S_n from its expected value,

$$\Delta_n(\omega) = \sup_t |S_n(\omega, t) - M_n(t)|,$$

to the process $\boldsymbol{\sigma} \cdot \mathbf{f}$ indexed by the random set

$$\mathcal{F}_{n\omega} = \{(f_1(\omega, t), \ldots, f_n(\omega, t)) : t \in T\}.$$

If we abbreviate the supremum of $|\boldsymbol{\sigma} \cdot \mathbf{f}|$ over $\mathcal{F}_{n\omega}$ to $L_n(\boldsymbol{\sigma}, \omega)$, the inequality becomes

$$\mathbb{P}\,\Phi(\Delta_n) \le \mathbb{P}\,\Phi(2L_n). \tag{7.1}$$

We bound the right-hand side by taking iterated expectations, initially conditioning on ω and averaging over $\boldsymbol{\sigma}$ with respect to the uniform distrbution $\mathbb{P}_\sigma$.

The chaining inequality from Theorem 3.5 bounds the conditional Ψ norm of L by

$$J_n(\omega) = 9 \int_0^{\delta_n(\omega)} \sqrt{\log D(x, \mathcal{F}_{n\omega})}\, dx, \qquad \text{where } \delta_n(\omega) = \sup_{\mathcal{F}_{n\omega}} |\mathbf{f}|.$$

Here, and throughout the section, the subscript 2 is omitted from the ℓ_2 norm $|\cdot|_2$; we will make no use of the ℓ_1 norm in this section. Written out more explicitly, the inequality that defines the Ψ norm becomes

$$\mathbb{P}_\sigma \exp\left(L_n(\boldsymbol{\sigma}, \omega)/J_n(\omega)\right)^2 \le 5. \tag{7.2}$$

Because J_n is a random variable, in general we cannot appeal directly to inequality (7.1) with $\Phi(x) = \exp(x^2/2J_n^2)$, to get some sort of bound for the Ψ norm of the