SECTION 6

Convex Hulls

Sometimes interesting random processes are expressible as convex combinations of more basic processes. For example, if $0 \le f_i \le 1$ for each i then the study of f_i reduces to the study of the random sets $\{\omega : s \le f_i(\omega, t)\}$, for $0 \le s \le 1$ and $t \in T$, by means of the representation

$$f_i(\omega, t) = \int_0^1 \{s \le f_i(\omega, t)\}\, ds.$$

More generally, starting from $f_i(\omega, t)$ indexed by T, we can construct new processes by averaging out over the parameter with respect to a probability measure Q on T:

$$f_i(\omega, Q) = \int f_i(\omega, t) Q(dt).$$

[This causes no measure-theoretic difficulties if there is a σ-field $\mathcal{T}$ on T such that f_i is jointly measurable in ω and t and Q is defined on $\mathcal{T}$.] Let us denote the corresponding process of sums by $S_n(\omega, Q)$, and its expectation by $M_n(Q)$. Because

$$|S_n(\omega, Q) - M_n(Q)| \le \int \sup_t |S_n(\omega, t) - M_n(t)| Q(dt),$$

it is easy to verify that

$$\sup_Q |S_n(\omega, Q) - M_n(Q)| = \sup_t |S_n(\omega, t) - M_n(t)|. \tag{6.1}$$

Some uniformity results for the processes indexed by probability measures on T follow trivially from uniformity results for processes indexed by T.

The operation of averaging out over t corresponds to the formation of convex combinations in $\mathbb{R}^n$. The vectors with coordinates $f_1(\omega, Q), \ldots, f_n(\omega, Q)$ all lie within the closed convex hull $\overline{co}(\mathcal{F}_\omega)$ of the set $\mathcal{F}_\omega$. The symmetrization analogue of the equality (6.1) is

$$\sup_{\overline{co}(\mathcal{F}_\omega)} |\boldsymbol{\sigma} \cdot \mathbf{f}| = \sup_{\mathcal{F}_\omega} |\boldsymbol{\sigma} \cdot \mathbf{f}|,$$

which suggests that there might be a connection between the packing numbers for $\mathcal{F}_\omega$ and the packing numbers for $\overline{co}(\mathcal{F}_\omega)$. A result of Dudley (1987) establishes such