SECTION 4

Packing and Covering in Euclidean Spaces

The maximal inequality from Theorem 3.6 will be useful only if we have suitable bounds for the packing numbers of the set $\mathcal{F}$. This section presents a method for finding such bounds, based on a geometric property that transforms calculation of packing numbers into a combinatorial exercise.

The combinatorial approach generalizes the concept of a Vapnik-Červonenkis class of sets. It identifies certain subsets of $\mathbb{R}^n$ that behave somewhat like compact sets of lower dimension; the bounds on the packing numbers grow geometrically, at a rate determined by the lower dimension. For comparison's sake, let us first establish the bound for genuinely lower dimensional sets.

(4.1) LEMMA. *Let $\mathcal{F}$ be a subset of a V dimensional affine subspace of $\mathbb{R}^n$. If $\mathcal{F}$ has finite diameter R, then*

$$D(\epsilon, \mathcal{F}) \leq \left(\frac{3R}{\epsilon}\right)^V \qquad \text{for } 0 < \epsilon \leq R.$$

PROOF. Because Euclidean distances are invariant under rotation, we may identify $\mathcal{F}$ with a subset of $\mathbb{R}^V$ for the purposes of calculating the packing number $D(\epsilon, \mathcal{F})$. Let $\mathbf{f}_1, \ldots, \mathbf{f}_m$ be points in $\mathcal{F}$ with $|\mathbf{f}_i - \mathbf{f}_j| > \epsilon$ for $i \neq j$. Let B_i be the (V-dimensional) ball of radius $\epsilon/2$ and center $\mathbf{f}_i$. These m balls are disjoint; they occupy a total volume of $m(\epsilon/2)^V \Gamma$, where Γ denotes the volume of a unit ball in $\mathbb{R}^V$. Each $\mathbf{f}_i$ lies within a distance R of $\mathbf{f}_1$; each B_i lies inside a ball of radius $3/2 R$ and center $\mathbf{f}_1$, a ball of volume $(3/2 R)^V \Gamma$. It follows that $m \leq (3R/\epsilon)^V$. □

A set of dimension V looks thin in $\mathbb{R}^n$. Even if projected down onto a subspace of $\mathbb{R}^n$ it will still look thin, if the subspace has dimension greater than V. One way to capture this idea, and thereby create a more general notion of a set being thin, is to think of how much of the space around any particular point can be occupied by