SECTION 3

Chaining

The main aim of the section is to derive a maximal inequality for the processes $\boldsymbol{\sigma} \cdot \mathbf{f}$, indexed by subsets of $\mathbb{R}^n$, in the form of an upper bound on the Ψ norm of $\sup_{\mathcal{F}} |\boldsymbol{\sigma} \cdot \mathbf{f}|$. [Remember that $\Psi(x) = {}^1\!/_5 \exp(x^2)$.] First we need a bound for the individual variables.

(3.1) LEMMA. *For each* $\mathbf{f}$ *in* $\mathbb{R}^n$, *the random variable* $\boldsymbol{\sigma} \cdot \mathbf{f}$ *has subgaussian tails, with Orlicz norm* $\|\boldsymbol{\sigma} \cdot \mathbf{f}\|_\Psi$ *less than* $2|\mathbf{f}|$.

PROOF. The argument has similarities to the randomization argument used in Section 2. Assume the probability space is a product space supporting independent $N(0,1)$ distributed random variables $g_1, \ldots, g_n$, all of which are independent of the sign variables $\sigma_1, \ldots, \sigma_n$. The absolute value of each g_i has expected value

$$\gamma = \mathbb{P}|N(0,1)| = \sqrt{2/\pi}.$$

By Jensen's inequality,

$$\begin{aligned} \mathbb{P}_\sigma \exp\Big(\sum_{i\le n} \sigma_i f_i / C\Big)^2 &= \mathbb{P}_\sigma \exp\Big(\sum_{i \le n} \sigma_i f_i \mathbb{P}_g |g_i| / \gamma C\Big)^2 \\ &\le \mathbb{P}_\sigma \mathbb{P}_g \exp\Big(\sum_{i\le n} \sigma_i |g_i| f_i / \gamma C\Big)^2. \end{aligned}$$

The absolute value of any symmetric random variable is independent of its sign. In particular, under $\mathbb{P}_\sigma \otimes \mathbb{P}_g$ the products $\sigma_1|g_1|, \ldots, \sigma_n|g_n|$ are independent $N(0,1)$ random variables. The last expected value has the form $\mathbb{P}\exp\big(N(0,\tau^2)^2\big)$, where the variance is given by

$$\tau^2 = \sum_{i \le n} (f_i/\gamma C)^2 = |\mathbf{f}|^2/\gamma^2 C^2.$$

Provided $\tau^2 < 1/2$, the expected value is finite and equals $(1 - 2|\mathbf{f}|^2/\gamma^2 C^2)^{-1}$. If we choose $C = 2|\mathbf{f}|$ this gives $\mathbb{P}\,\Psi(\boldsymbol{\sigma} \cdot \mathbf{f}/C) \le 1$, as required. □