Moduli spaces of twisted sheaves on a projective variety

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Dedicated to Masaki Maruyama on the occasion of his 60th birthday

Appendix by Daniel Huybrechts and Paolo Stellari

§0. Introduction

Let $X$ be a smooth projective variety over $\mathbb{C}$. Let $\alpha := \{\alpha_{ijk} \in H^0(U_i \cap U_j \cap U_k, O_X)\}$ be a 2-cocycle representing a torsion class $[\alpha] \in H^2(X, O_X^*)$. An $\alpha$-twisted sheaf $E := \{(E_i, \varphi_{ij})\}$ is a collection of sheaves $E_i$ on $U_i$ and isomorphisms $\varphi_{ij} : E_i|_{U_i \cap U_j} \to E_j|_{U_i \cap U_j}$ such that $\varphi_{ii} = \text{id}_{E_i}$, $\varphi_{ji} = \varphi_{ij}^{-1}$ and $\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} = \alpha_{ijk} \text{id}_{E_i}$. We assume that there is a locally free $\alpha$-twisted sheaf, that is, $\alpha$ gives an element of the Brauer group $\text{Br}(X)$. A twisted sheaf naturally appears if we consider a non-fine moduli space $M$ of the usual stable sheaves on $X$. Indeed the transition functions of the local universal families satisfy the patching condition up to the multiplication by constants and gives a twisted sheaf. If the patching condition is satisfied, i.e., the moduli space $M$ is fine, than the universal family defines an integral functor on the bounded derived categories of coherent sheaves $D(M) \to D(X)$. Assume that $X$ is a $K3$ surface and $\dim M = \dim X$. Then Mukai, Orlov and Bridgeland showed that the integral functor is the Fourier-Mukai functor, i.e., it is an equivalence of the categories. In his thesis [C2], Căldăraşu studied the category of twisted sheaves and its bounded derived category. In particular, he generalized Mukai, Orlov and Bridgeland's results on the Fourier-Mukai transforms to non-fine moduli spaces on a $K3$ surface. For the usual derived category, Orlov [Or] showed that the equivalence class is described in terms of the Hodge structure of the Mukai lattice. Căldăraşu conjectured that a similar result also holds for the derived

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