# Remarks and Historical Notes

*Chapter I*

Sections 1–6

This is all basic set theory. See the "Historical Notes" section of Jech (1978) for historical and bibliographical details.

Section 7

Theorems 7.1 and 7.2 are due (independently) to Mostowski and Shepherdson and appeared in Mostowski (1949).

Section 8

Theorems 8.1 and 8.2 are due to Montague and Lévy. See Montague (1961) and Lévy (1960). The Lévy Hierarchy was introduced in Lévy (1965). The concept of absoluteness is due to Gödel, who in principle proved 8.3.

Section 9

There are various (essentially equivalent) ways of defining a formal "language of set theory" within set theory. The precise definition chosen is our own, as is all of the development in this section. The Basic Set Theory (BS) has been formulated independently by various authors, among them Gandy, Jensen, and ourselves. The first published account of this seems to be in Gandy (1974).

Section 10

Again the material of this section is our own version of well-known, essentially "folklore" material.

Section 11

Admissible sets were first studied by S. Kripke and R. Platek. See Kripke (1964) and Platek (1966). Most of the results in this section now have the status of "folklore". For further details on admissible sets consult Barwise (1975).

*Chapter II*

Section 1

The notion of constructibility is due to K. Gödel. In Gödel (1939), the sets $L_\alpha$ were defined and used to define $L$. In his famous 1940 monograph (the first ever "book on $L$"), Gödel defined $L$ using a collection of definability functions. All of the material in this section is due to Gödel.

Section 2

The material in this section was known implicitly to Gödel. The first explicit proof that "$L$ is $\Sigma_1$" appeared in Karp (1967). The proof given here is our own version.

Section 3

The validity of $AC$ in $L$ was announced in Gödel (1938), and a proof was given in Gödel (1940). The present proof is our own version of Gödel's original argument.

Section 4

Theorem 4.3 is due to Shepherdson (1951, 1952, 1953).

Section 5

Again these results were all known to Gödel in one form or another, with the proof given here our own version.

Section 6

The material in this section is due mainly to R.B. Jensen. Various sets of notes on the results were in circulation for several years in the early to mid 1960s, but the material was never published until the appearance of Jensen (1972) and Devlin (1973).

Section 7

The remarks for Section 6 apply.

Exercises

1. The primitive recursive set functions were first studied in depth by Jensen and Karp (1971).
2. Relative constructibility was first investigated by Hajnal (1956) and Lévy (1957).
3. This was proved by various people, ourselves included.
4. Hajnal (1956).
5. Jensen (unpublished).

## *Chapter III*

### Section 1

The Souslin Problem was proposed in M. Souslin (1920). The first systematic investigation of trees was carried out by D. Kurepa (1935). This paper includes Theorems 1.1 and 1.4. Both of these were later rediscovered by other authors, among them F.B. Jones (Theorem 1.1) and E. Miller (Theorem 1.4). Theorem 1.5 is due to Jensen, who announced the result in Jensen (1968). The first construction of a model of set theory in which there are no Souslin trees is due to Solovay and Tennenbaum (1971). See Jech (1978) for further details.

### Section 2

The Kurepa Hypothesis was first formulated in Kurepa (1935). The proof that $KH$ is equivalent to the existence of a Kurepa tree is also due to Kurepa himself. The construction of a Kurepa tree from $V = L$ is due to Solovay (unpublished), shortly after Jensen's construction of a Souslin tree in $L$. The first construction of a model of set theory in which there are no Kurepa trees is due to Silver (1971). Silver started with a model in which there is an inaccessible cardinal. Solovay observed that since his construction of a Kurepa tree in $L$ works (with minor changes) for any universe $L[A]$ where $A \subseteq \omega_1$ (in $L[A]$), Silver's assumption of the consistency of an inaccessible cardinal is necessary. For further details on Kurepa trees, consult Jech (1978).

### Section 3

Fodor's Theorem (Theorem 3.1) appeared in Fodor (1956). Lemma 3.4 is due to K. Kunen. Everything else in this section is due to Jensen.

### Exercises

1D. This is due to Baumgartner (unpublished).
2. Solovay (unpublished).
3A–3D. These are due to Devlin (1979a).
4. Jensen (unpublished).

## *Chapter IV*

### All Sections

Practically all of this is due to Jensen. The results were proved and circulated privately around 1967/68, but did not appear in print until the appearance of Jensen (1972) and Devlin (1973). See the notes for Chapter VI for more details along this line. Theorem 2.11 was first proved by J. Gregory (1976) using a direct argument. The proof given here, using 2.5 through 2.10, is a distillation of subsequent observations of Jensen, Gregory, Laver, and ourselves.

Exercises

1 A. Aronszajn.
1 B. Specker.
1 C. Jensen.
2. Jensen.
3. This observation was used by Rowbottom and Lévy (independently) to obtain a model of set theory in which there is a Kurepa tree.
4. Jensen.
5. Jensen.
6. The consistency result mentioned here is due to Solovay (unpublished).
7, 8. These results were formulated by Shelah, and sharpen previous results of Gregory, Jensen, Laver, and ourselves.

*Chapter V*

Section 1

Drake (1974) and Jech (1978) supply all of the relevant history, etc.

Section 2

In 1964, F. Rowbottom proved, in his PhD thesis (Wisconsin) that if there is a measurable cardinal ($\kappa(\omega_1)$ suffices), then $\mathscr{P}^L(\omega)$ is countable. This finally appeared in Rowbottom (1971). Rowbottom used the method of indiscernibles to prove this result. In the same year, H. Gaifman used the method of iterated ultrapowers to obtain the conclusion of Theorem 2.11, and with it various consequences, including Rowbottom's result, from the existence of a measurable cardinal. See Gaifman (1964). Solovay extracted $0^\#$ from the proof, and proved various result about $0^\#$. See Solovay (1967). In his PhD thesis (Berkeley, 1966), J. Silver subsequently obtained all of Gaifman's and Solovay's result using the method of indiscernibles, starting from the existence of $\kappa(\omega_1)$. Silver's work was finally published in Silver (1971).

The methods used throughout this section are essentially those developed by Silver (with one or two ideas from Rowbottom). See Jech (1978) for further details concerning $0^\#$.

Section 3

This is due to Solovay, who also proved that $0^\#$ is $\Delta^1_3$-definable over the integers.

Section 4

The main result of this section are due to K. Kunen. The proofs given are due to Silver.

Section 5

The Covering Lemma is due to Jensen, and appeared in Devlin and Jensen (1975). The proof given here is a somewhat simpler version due to M. Magidor.

Exercises

1 A. Silver.
1 B. Various people.
1 C. Harrington and Shelah.
2. Solovay.
3. Various people.
4 B, 4 C. Bukovski.

*Chapter VI*

All Sections

Everything in this chapter is due to Jensen. (Exception: Theorem 6.1′ is due to Beller and Litman (1980), following Jensen's proofs.) Jensen first worked out a "fine structure theory" for the $L_\alpha$-hierarchy in the early to mid 1960s. His results were circulated in the form of a number of increasingly bulky sets of handwritten notes. Then, in 1970 he defined the $J_\alpha$-hierarchy and reworked the fine structure theory for this hierarchy. Working from his notes, we then wrote three sets of (handwritten) notes which for a while became the "standard text" in the area. These notes were subsequently turned into the book Devlin (1973). In the meantime, Jensen (1972) came out, covering some of the same material. Until the appearance of this volume, these were essentially the only sources for this material.

Exercises

1. Jensen. See Devlin and Jensen (1975).
2. Prikry and Solovay (1975).
2D. Starting with a model with a Mahlo cardinal, Shelah has constructed a model of set theory in which every stationary subset of $\omega_2$ consisting of $\omega$-cofinal ordinals contains a closed set of type $\omega_1$.
4. Beller and Litman. (Unpublished in this form.)

*Chapter VII*

All Sections

Theorem 1.2 is due to Jensen (1972). Theorem 1.2′ is due to Beller and Litman (1980). The rest of the chapter is due to Jensen.

Exercises

1. Devlin.
2. Shelah.
3 A. Erdos, Hajnal and Milner.
3 B. Prikry, Jensen (independently). More recently, S. Todorcevic has proved that $KH(\kappa,\kappa)$ for $\mathrm{cf}(\kappa)=\omega$ follows from $\Box_\kappa$.
4. Beller and Litman.

## *Chapter VIII*

### Section 1

Theorem 1.7 and its proof are due to R.L. Vaught. (See Morley & Vaught (1962).)

### Section 2

Everything in this section is due to Jensen.

### Section 3

Theorem 3.1 is due to Jensen. In his original proof, Jensen made use of some techniques developed by Keisler involving omitting types arguments. The proof given, using a $\Delta$-system approach was later worked out by Donder. The construction of the $\mathscr{M}$-complex is effectively the same as in the original Jensen proof.

### Section 4

Simplified morasses were developed by Velleman (see Velleman (1984, 1984a)), following some work on morasses by Shelah and Stanley (1983). The construction of a simplified morass from a standard morass is due to Donder (1983).

Another simplified morass-like structure is due to Silver, and is known as *W*. For details of this see Kanamori (1982).

The application of both the standard morass and the simplified morass given here uses the morass only as an "indexing system". Many of the other uses of morasses employ diagonalisation techniques using $\diamondsuit$-like properties that can be incorporated into morasses. Exercise 4 provides one example. Others can be found in the references given above.

### Section 5

At present only reference to gap-*n* morasses is Stanley (1975).

## *Chapter IX*

### All Sections

The notion of a machine was worked out by Silver to provide an alternative to the fine structure theory for proving $\square_\kappa$ and similar results. The construction of a machine given here, though essentially that of Silver, owes much to an unpublished account of the subject written by A. Litman. The proof of $\square$ using a machine, whilst again Silver's proof at heart, follows closely the version given in Beller and Litman (1980).