Algebra structures coming from tangles

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Abstract

In an attempt to find analogues, for higher relative commutants, of the Bose-Mesner algebra structure on the second relative commutant of a spin-model subfactor, we find that there do indeed exist other algebra structures on the higher relative commutants of any subfactor planar algebra which are induced by the action of tangles; in fact, we show they can only arise in the obvious fashion.

1 Introduction

It is a pleasure and an honour to be permitted to contribute this small note in a volume being brought out in honour of Vaughan Jones. We hope this note will provide one further iota of evidence of the power and beauty of the planar algebra formalism ([J]) of Jones.

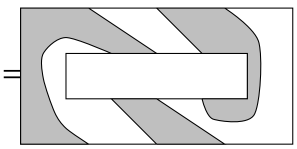
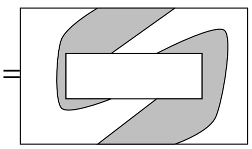
It is known - see [JMN] - that if $N \subset M$ is the *spin model subfactor* associated to a complex Hadamard matrix in $M_m(\mathbb{C})$, then the second relative commutant $N' \cap M_1$ is endowed with a natural structure of a Bose-Mesner algebra. But nothing along those lines seems to be known for the higher relative commutants $N' \cap M_n, n > 1$. On the other hand, it is a fact that if $\mathcal{F} : N' \cap M_1 \to M' \cap M_2$ is the subfactor analogue of the Fourier transform, then $N' \cap M_1 \subset M_m(\mathbb{C})$ and $a \circ b = \mathcal{F}^{-1}(\mathcal{F}(a)\mathcal{F}(b)) \; \forall a, b \in N' \cap M_1$, where $a \circ b$ denotes the Schur product of the matrices a and b. Equivalently, $a \circ b = Z_C(a \otimes b)$ where C denotes the tangle that is sometimes referred to (albeit inappropriately) as the 'comultiplication tangle'.

We shall adopt the convention of denoting a k-tangle T by $T^k_{k_1,\cdots,k_b}$ if it is important to show that it has b internal discs respectively of colours $k_1, \cdots, k_b$ - so iff $b = 0$, T has no internal discs and in that case, $Z_T : \mathbb{C} \to P_k$; in particular, $Z_{1^k}(1)$ is the multiplicative identity of P_k. We show, conversely, that if P is a subfactor planar algebra, and if there exist k-tangles U^k and $P^k_{k,k}$, which define a 'unit tangle' and a 'product tangle' on the space P_k such that $Z_P(Z_U(1) \otimes x) = Z_P(x \otimes Z_U(1)) = x$, and if R denotes the '1-click' rotation k-tangle (so that $Z_R : P_{k+} \to P_{k_-}$), then the tangles U and P are given, respectively, by $U = R^{-i} \circ 1$ and $P = R^{-i} \circ M_k \circ (R^i, R^i)$. (A more precise version of the result described above may be found in Theorem 1.)

2 All algebra structures coming from tangle actions

Definition 1. **(Rotation tangles of colour** $(k, \varepsilon), k \geq 1$**:)** For $k \geq 1, \varepsilon \in \{\pm\}$, define $R^i_{(k,\varepsilon)}$, $1 \leq i \leq 2k$, to be the annular tangle (i.e., with one internal box) in which the internal box has color (k, ε), and the j-th marked point of the external box is connected to the $(i + j)$-th marked point of the internal box for every $1 \leq j \leq 2k$ (with addition modulo $2k$). Our convention is that the points are numbered clockwise with the $*$-interval being between $2k$ and 1. (It should be noted that the external box of, and hence the tangle, $R^i_{(k,\varepsilon)}$ has colour $(k, \pm\varepsilon)$ according as i is even or odd.)

The tangles $R^1_{(3,+)}$ and $R^3_{(2,-)}$ are illustrated as in the figure below:[1]

$R^1_{(3,+)} =$  $R^3_{(2,-)} =$ 

In this section we will prove following result.

Theorem 1. *Suppose P is a non-trivial (i.e., the modulus $\delta > 1$) subfactor planar algebra and T is a tangle of color (k, ε) that contains two internal boxes, both also of color (k, ε), where $k \neq 2$. Suppose there exist a Temperley-Lieb tangle $U^{(k,\varepsilon)}$ of the same color (k, ε) such that $Z_T(Z_{U^{(k,\varepsilon)}}(1) \otimes x) = Z_T(x \otimes Z_{U^{(k,\varepsilon)}}(1)) = x$ for all x in $P_{(k,\varepsilon)}$. Then there exist $1 \leq i \leq 2k$, such that $T = R^{2k-i}_c \circ (M_c \circ (R^i_{(k,\varepsilon)}, R^i_{(k,\varepsilon)}))$ and $U^{(k,\varepsilon)} = R^{2k-i}_c \circ 1^c$, where c is (k, ε) or $(k, -\varepsilon)$ according as i is even or odd . Consequently Z_T gives an automatically associative product on $P_{(k,\varepsilon)}$.*[2]

Remark 1. Suppose P is a non-trivial subfactor planar algebra. Then any two Temperley Lieb elements in $P_{(k,\varepsilon)}$, which have no loops in them, are linearly independent. This is an immediate consequence of the Cauchy Schwarz inequality - as was kindly pointed out to us by Vaughan Jones.

For proving Theorem 1, we will first prove the following Lemma.

Lemma. *If P is a subfactor planar algebra with modulus $\delta > 1$, and T is an annular tangle of color (k, ε) with internal box also of color (k, ε) such that $Z_T = id_{P_{(k,\varepsilon)}}$, and if $k \neq 2$, then T is the identity tangle $I^{(k,\varepsilon)}_{(k,\varepsilon)}$.*

Proof. If T had a contractible loop, then $Z_{1^{(k,\varepsilon)}}(1) = Z_T(Z_{1^{(k,\varepsilon)}}(1)) = Z_{T\circ 1^{(k,\varepsilon)}}(1) = \delta^j Z_L(1)$, where $j \geq 1$ and L is the Temperley Lieb tangle without loops obtained by removing all loops from $T \circ 1^{(k,\varepsilon)}$. By Remark 1 and $\delta > 1$, this is not possible. Therefore T cannot have any contractible loops.

Suppose, if possible, that T has a cap on an internal box which joins marked point u to marked point v. Let us fix any Temperley Lieb tangle L of color (k, ε) without loops such that the marked point u is connected to the marked point v. Then $Z_L(1) = Z_T(Z_L(1)) = Z_{T\circ L}(1) = \delta^j Z_{L_1}(1)$, where $j \geq 1$ and L_1 is the Temperley Lieb tangle without loops obtained by removing all loops from $T \circ L$; but by remark 1 this is not possible. Therefore T cannot have any cap on the internal box.

[1] Here and elsewhere when figures are drawn, it is always understood that the $*$-interval of any box contains the left vertical bounding edge.

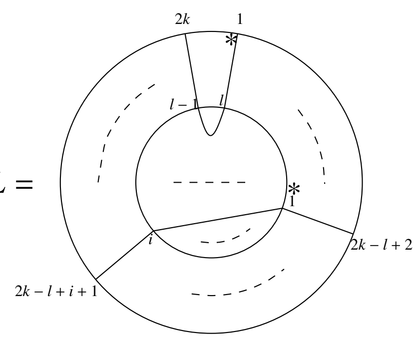
[2] We use the notation of [KS] - M_c (resp., 1^c) denotes the multiplication (resp., unit) tangle of colour c.

Note that if internal and external boxes are of the same color and if the internal box does not have any caps, then the external box can also not have any caps. So T does not have any cap on the external box either.

For the case of colors $(0, \varepsilon)$, if T has j loops, we have $Z_{1^{(0,\varepsilon)}}(1) = Z_T(Z_{1^{(0,\varepsilon)}}(1)) = \delta^j Z_{1^{(0,\ell\varepsilon)}}(1)$, where $j \geq 1$, which is clearly impossible since $\delta > 1$.

The case of $(1, \varepsilon)$ is proved exactly as in the case of $(0, \varepsilon)$, so assume $k > 2$. Since T has no caps on external or internal box and has no contractible loops, it follows that there exist $1 \leq l \leq 2k$ such that the j-th marked point on the external box is connected to the $j + l - 1$-th marked point on the internal box. If we prove $l = 1$, this would complete the proof of the lemma.

Now, the internal and external box have the same color, and the point marked 1 on the external box is connected to the point marked l on the internal box. Clearly, l must be odd. Suppose, if possible, that $l \geq 3$. Since $k > 2$, there exist at least 3 even numbers in $\{1, 2, \cdots, 2k\}$. So we can choose an even number $i \neq l - 1$ such that $1 \leq i < 2k$. Then fix one Temperley Lieb tangle L of color (k, ε), without loops subject only to the condition that 1 is connected to i and $l - 1$ is connected to l. Then by hypothesis on T, we have $Z_L(1) = Z_T(Z_L(1)) = Z_{T \circ L}(1)$. Here $T \circ L$ also a Temperley Lieb tangle without loops, so by Remark 1, we have $L = T \circ L$.

$T \circ L =$ 

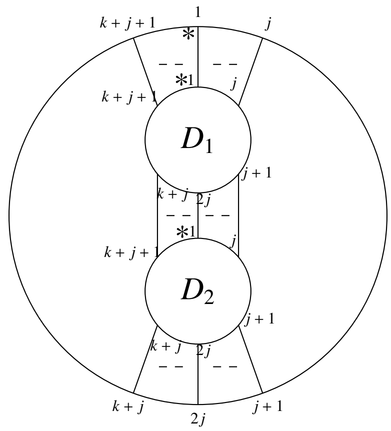
Thus, the marked point 1 of the external box is connected to the marked point $2k$ (respectively i) of the internal box in $T \circ L$ (respectively in L), and $i \neq 2k$. Therefore $l \geq 3$ cannot be true. So l must be 1; and the proof of the lemma is complete.

□

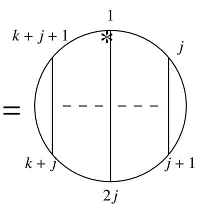
Proof. (of Theorem 1): The hypothesis $Z_T(Z_{U^{(k,\varepsilon)}}(1) \otimes x) = Z_T(x \otimes Z_{U^{(k,\varepsilon)}}(1)) = x$ implies $Z_{T \circ_{D_i} U^{(k,\varepsilon)}} = id_{P_{(k,\varepsilon)}}$, for $i = 1, 2$. So our Lemma implies $T \circ_{D_i} U^{(k,\varepsilon)} = I^{(k,\varepsilon)}_{(k,\varepsilon)}$ for $i = 1, 2$. For colors $(0, \varepsilon)$, $(1, \varepsilon)$ this theorem is clear. So assume $k > 2$. Note that the above conclusion $T \circ_{D_i} U^{(k,\varepsilon)} = I^{(k,\varepsilon)}_{(k,\varepsilon)}, i = 1, 2$ implies that T cannot have either loops or caps on internal or external boxes, and $U^{(k,\varepsilon)}$ can also not contain any loop. Also note that $T \circ_{D_i} U^{(k,\varepsilon)} = I^{(k,\varepsilon)}_{(k,\varepsilon)}$ for $i = 1, 2$ implies that every marked point of the external box must be connected to a marked point *with the same label* of either D_1 or D_2.

Let l, m, n denote the number of strings which join the external box to the first box, the first box to the second box, and the second box to the external box, respectively. Then, $l + m = m + n = l + n = 2k$ and hence $l = n = m = k$. Since there are no caps on any box, consecutively marked points (modulo $2k$) of external box must join consecutive marked points of the first (resp., second) internal box; and a similar remark applies for strings joining the two internal boxes.

Case 1*:* If marked point 1 of external box of T is connected to marked point 1 of D_1, then there exists $1 \leq j \leq k$ such that T looks as in the following diagram:

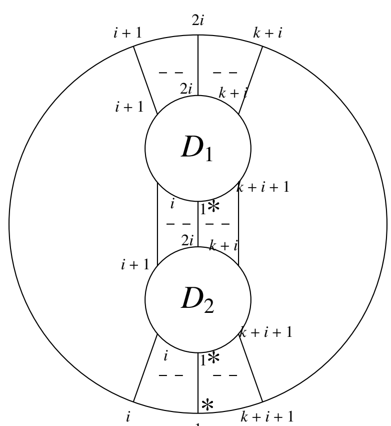


If $k+j$ is even, then $T = R^{k-j}_{(k,\varepsilon)} \circ (M_{(k,\varepsilon)} \circ (R^{k+j}_{(k,\varepsilon)}, R^{k+j}_{(k,\varepsilon)}))$, and if $k+j$ is odd, then $T = R^{k-j}_{(k,-\varepsilon)} \circ (M_{(k,-\varepsilon)} \circ (R^{k+j}_{(k,\varepsilon)}, R^{k+j}_{(k,\varepsilon)}))$. Note that $T \circ_{D_1} U^{(k,\varepsilon)} = I^{(k,\varepsilon)}_{(k,\varepsilon)}$ implies $U^{(k,\varepsilon)}$ must be as indicated below:

$U^{(k,\varepsilon)} =$ 

So $U^{(k,\varepsilon)} = R^{k-j}_{(k,\varepsilon)} \circ 1^{(k,\varepsilon)}$ if $k+j$ is even, otherwise $U^{(k,\varepsilon)} = R^{k-j}_{(k,-\varepsilon)} \circ 1^{(k,-\varepsilon)}$. So $k+j$ is the i as in the theorem.

Case 2*:* If marked point 1 of the external box of T is connected to marked point 1 of D_2, then there exists $1 \leq i \leq k$ such that T looks as in the following diagram:



If i is even, then $T = R^{2k-i}_{(k,\varepsilon)} \circ (M_{(k,\varepsilon)} \circ (R^{i}_{(k,\varepsilon)}, R^{i}_{(k,\varepsilon)}))$, otherwise i is odd and $T = R^{2k-i}_{(k,-\varepsilon)} \circ (M_{(k,-\varepsilon)} \circ (R^{i}_{(k,\varepsilon)}, R^{i}_{(k,\varepsilon)}))$. Note that in the same way as above, $U^{(k,\varepsilon)} = R^{2k-i}_{(k,\varepsilon)} \circ 1^{(k,\varepsilon)}$ if i is even, otherwise $U^{(k,\varepsilon)} = R^{2k-i}_{(k,-\varepsilon)} \circ 1^{(k,-\varepsilon)}$.

□

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