UNIQUENESS PROPERTIES OF DIFFUSION PROCESSES

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Abstract. We review recent results on the uniqueness of solutions of the diffusion equation

$$\partial\psi_t/\partial t + H\psi_t = 0$$

where H is a strictly elliptic, symmetric, second-order operator on an open subset Ω of $\mathbf{R}^d$. In particular we discuss L_1-uniqueness, the existence of a unique continuous solution on $L_1(\Omega)$, and Markov uniqueness, the existence of a unique submarkovian solution on the spaces $L_p(\Omega)$. We give various criteria for uniqueness in terms of capacity estimates and the Riemannian geometry associated with H.

1. Introduction

Let Ω be a connected open subset of $\mathbf{R}^d$ and H a second-order, formally symmetric, elliptic operator on the domain $D(H) = C_c^\infty(\Omega)$. The operator H is defined to be L_1-unique if it has a unique L_1-closed extension H_1 which generates a strongly continuous semigroup T on $L_1(\Omega)$. This is equivalent to the parabolic diffusion equation

$$\partial\psi_t/\partial t + H\psi_t = 0 \tag{1}$$

having a unique continuous weak solution $t > 0 \mapsto \psi_t = T_t\psi_0 \in L_1(\Omega)$ for each $\psi_0 \in C_c^\infty(\Omega)$. The ellipticity property of H, which will be specified in more detail below, implies that H is L_1-dissipative and consequently L_1-closable. Then it follows by an extension of the Lumer–Phillips theorem that H is L_1-unique if and only if $H_1 = \overline{H}^{\,1}$, the L_1-closure of H. Our primary intention is to review recent results which establish alternative criteria for L_1-uniqueness of H.

Evolution equations of the type (1) occur in many applications since they typically describe a diffusion process. Positive normalized L_1-solutions are of particular interest since they can be interpreted as probability distributions. Specifically the solutions with a probabilistic interpretation are given by L_1-continuous semigroups T satisfying

- **positivity:** $T_t\varphi \geq 0$ for all positive $\varphi \in L_1(\Omega)$ and $t > 0$, and
- **conservation of probability:** $\|T_t\varphi\|_1 = \|\varphi\|_1$ for all positive $\varphi \in L_1(\Omega)$ and $t > 0$.

The latter property is in fact closely related to the L_1-uniqueness. This will be discussed in Section 4.

The difficulty in establishing uniqueness properties is that they are not generally true: the symmetric diffusion process described by H is not expected to be uniquely determined. There are two distinct reasons for lack of uniqueness, one local and one global. If the diffusion can either reach

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the (local) boundary $\partial\Omega = \overline{\Omega}\backslash\Omega$ or can spread to infinity in a finite time then the subsequent properties of the evolution depend on the specification of boundary conditions. But the process will not reach $\partial\Omega$ if the diffusion slows sufficiently near the boundary. Alternatively, the diffusion will not spread to infinity in a finite time unless it accelerates sufficiently as it recedes. Although these obstructions to uniqueness appear to be of a similar nature, both relate to the accessibility of the relevant boundary, they do differ in nature. To be more precise we must be more specific about the structure of the operator H.

Throughout the sequel we assume that H is a second-order operator in divergence form,

$$H = -\sum_{k,l=1}^{d} \partial_k \, c_{kl} \, \partial_l \tag{2}$$

where the $c_{kl} = c_{lk}$ are real-valued $W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$-functions, and the matrix $C = (c_{kl})$ is strictly elliptic, i.e. $C(x) > 0$ for all $x \in \Omega$. Here $W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$ denotes the space of restrictions to Ω of functions in $W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}^d)$. The conditions on the c_{kl} ensure they extend continuously to the boundary $\partial\Omega$ and the extensions are locally bounded. Moreover, it follows from these assumptions that H is locally strongly elliptic, i.e. for each relatively compact subset V of Ω there are $\mu_V, \lambda_V > 0$ such that $\mu_V I \leq C(x) \leq \lambda_V I$ for all $x \in V$. Nevertheless the coefficients can degenerate on the boundary. In particular one can have $c_{kl}(x) \to 0$ as $x \to \partial\Omega$ or $c_{kl}(x) \to \infty$ as $|x| \to \infty$.

The operator H is positive and symmetric on $L_2(\Omega)$ and the corresponding quadratic form h is given by $D(h) = C_c^\infty(\Omega)$ and

$$h(\varphi) = \sum_{k,l=1}^{d} (\partial_k\varphi, c_{kl} \, \partial_l\varphi) \; .$$

The form is closable and its closure $h_D = \overline{h}$ determines a positive self-adjoint extension H_D of H, the Friedrichs' extension (see, for example, [8], Chapter VI). We have introduced the notation H_D since this extension corresponds to Dirichlet conditions on the boundary $\partial\Omega$. The closure h_D is a Dirichlet form and consequently the extension H_D is submarkovian, i.e. it generates a strongly continuous self-adjoint contraction semigroup S on $L_2(\Omega)$ satisfying $0 \leq S_t\varphi \leq 1$ for all $\varphi \in L_2(\Omega)$ with $0 \leq \varphi \leq 1$ (for details on Dirichlet forms and submarkovian semigroups see [2] [5] [10]). The semigroup S extends from $L_2(\Omega) \cap L_p(\Omega)$ to a positive contraction semigroup $S^{(p)}$ on each of the spaces $L_p(\Omega)$, $p \in [1, \infty]$, and the generator H_p of $S^{(p)}$ is an L_p-closed extension of H. Therefore H has a submarkovian extension and also a generator extension on each of the L_p-spaces. In particular this establishes that H has an L_1-generator extension and that (1) has a positive L_1-continuous solution $\psi_t = S_t\psi$ satisfying $\|S_t\psi\|_1 \leq \|\psi\|_1$ for all $\psi \in L_1(\Omega)$ and $t > 0$.

The foregoing discussion raises the question of establishing criteria for H to be Markov unique, i.e. for H to have a unique submarkovian extension. Our secondary intention is to review the characterization of Markov uniqueness. Each submarkovian extension determines a continuous semigroup on $L_1(\Omega)$ and the generator of this semigroup is an L_1-extension of H. It follows readily that distinct submarkovian extensions determine distinct L_1-extensions. Therefore *L_1-uniqueness of H implies Markov uniqueness.* Markov uniqueness is in general a strictly weaker property than L_1-uniqueness but we will describe various conditions which ensure they are equivalent.

2. Inaccessibility and Negligibility

The diffusion process described by H can be independent of boundary effects for two different reasons. Either the diffusion never reaches the boundary or the boundary is sufficiently insignificant that it does not influence the process. There are also two distinct factors which govern the independence.

The first factor is the geometry inherent to the process. The Riemannian geometry associated with the operator H is determined by the distance $d(\,\cdot\,;\,\cdot\,)$ corresponding to the metric C^{-1} on Ω. This distance is naturally suited to measurement of the diffusion. It can be defined in various equivalent ways but in particular by

$$d(x\,;y) = \sup\{\psi(x) - \psi(y) : \psi \in W^{1,\infty}_{\mathrm{loc}}(\Omega)\,,\,\Gamma(\psi) \leq 1\} \tag{3}$$

for all $x, y \in \Omega$ where Γ, the *carré du champ* of H, denotes the positive map

$$\varphi \in W^{1,2}_{\mathrm{loc}}(\Omega) \mapsto \Gamma(\varphi) = \sum_{i,j=1}^{d} c_{ij}(\partial_i\varphi)(\partial_j\varphi) \in L_{1,\mathrm{loc}}(\Omega)\ . \tag{4}$$

Since Ω is connected and $C > 0$ it follows that $d(x\,;y)$ is finite for all $x, y \in \Omega$ but one can have $d(x\,;y) \to \infty$ as x, or y, tends to the boundary $\partial\Omega$. Throughout the sequel we choose coordinates such that $0 \in \Omega$ and denote the Riemannian distance to the origin by ρ. Thus $\rho(x) = d(x\,;0)$ for all $x \in \Omega$. It is to be expected that the asymptotic behaviour of ρ gives a measure of accessibility of the 'boundary at infinity'. If $\rho(x) \to \infty$ as $x \to \infty$ then it is plausible that the diffusion will take an infinite time to reach infinity which is one interpretation of the statement that the boundary is inaccessible. It is the most elementary condition which could possibly rule out ambiguity due to the boundary at infinity.

The asymptotic behaviour can also be expressed in terms of the Riemannian balls $B(r) = \{x \in \Omega : \rho(x) < r\}$. One readily establishes that *$\rho(x) \to \infty$ as $x \to \infty$ if and only if the balls $B(r)$ are bounded subsets of Ω for all $r > 0$.* The boundedness of the balls $B(r)$ is the principal asymptotic feature of importance for Markov uniqueness (see Theorem 3.3). The situation for L_1-uniqueness is slightly different. Then the rate of growth of the volume (Lebesgue measure) $|B(r)|$ of the balls as $r \to \infty$ is also crucial. It is a measure of the available volume for the diffusion to spread. The growth bounds $|B(r)| \leq a\,\exp(b\,r^2)$, for all $r > 0$, are sufficient for the equivalence of Markov uniqueness and L_1-uniqueness (see Theorem 4.3).

Both of the foregoing geometric features are related to the asymptotic growth of the coefficients c_{kl} of H. For example if $\Omega = \mathbf{R}^d$ and $C(x) = c(|x|)I$ with $c > 0$ then $\rho(x) = \int_0^{|x|} c^{-1/2}$. Thus if $c(|x|) \sim x^2(\log|x|)^\alpha$ as $x \to \infty$ with $\alpha \geq 0$ then $\rho(x) \sim (\log|x|)^{1-\alpha/2}$ as $x \to \infty$. Moreover, there are $a, b > 0$ such that $|B(r)| \leq a \exp(b\, r^{2/(2-\alpha)})$ for all $r > 0$. Therefore $\rho(x) \to \infty$ as $x \to \infty$ if and only if $\alpha < 2$ and the bounds $|B(r)| \leq a \exp(b\, r^2)$ are valid if and only if $\alpha \leq 1$. But if $d = 1$ then Example 4.2 in [12] gives an H with $c(|x|) \sim x^2(\log|x|)(\log(\log|x|))$ as $x \to \infty$ which is not L_1-unique. In this example the balls $B(r)$ are bounded and for each $\varepsilon > 0$ one has growth bounds $|B(r)| \leq a \exp(b\, r^{2+\varepsilon})$ for all $r > 0$. We will conclude that the growth $c(|x|) \sim x^2(\log|x|)$ corresponding to the bounds $|B(r)| \leq a \exp(b\, r^2)$, is essentially the maximal growth for L_1-uniqueness and the growth $c(|x|) \sim x^2(\log|x|)^2$ is essentially the maximal growth for Markov uniqueness.

The second factor which influences uniqueness properties is the accessibility or inaccessibility of the boundary $\partial\Omega$. Clearly this is influenced by the rate of diffusion near the boundary and that is again determined by the magnitude of the coefficients. But the nature of the boundary is also clearly important. For example, one would expect the reflection properties of a smooth boundary to be quite different to those of a rough, or fractal, boundary. The relevant property to assess the effect of the boundary is its capacity, measured in a suitable sense by the operator H. The notion of capacity originated in electrostatics as a measure of the charge necessary on a closed surface to give a prescribed potential in the interior. There are several classical ways of introducing the capacity but it is essentially a property of the Laplacian. Therefore it is not surprising that it also enters the theory of Brownian motion. In the latter context the sets with capacity zero correspond to the sets which are negligible for the motion. Analogously the uniqueness of the diffusion process is closely linked with the boundary having capacity zero. To make this relation precise we introduce the capacity corresponding as follows. Let A be a general subset of $\overline{\Omega}$ then its capacity, relative to H, is defined by

$$\operatorname{cap}(A) = \inf\{h_N(\psi) + \|\psi\|_2^2\}$$

where the infimum is taken over those $\psi \in W^{1,2}_{\rm loc}(\Omega)$ for which $0 \leq \psi \leq 1$ and there exists an open set $U \subset \mathbf{R}^d$ such that $U \supseteq A$ and $\psi = 1$ on $U \cap \Omega$. Here the quadratic form h_N is defined by

$$D(h_N) = \{\varphi \in W^{1,2}_{\rm loc}(\Omega) : \Gamma(\varphi) + \varphi^2 \in L_1(\Omega)\}$$

and

$$h_N(\varphi) = \int_\Omega \Gamma(\varphi)\ .$$

The form h_N is a Dirichlet form and the associated operator is an extension H_N of H which corresponds to Neumann boundary conditions on $\partial\Omega$ at least if the boundary is smooth. This definition of the capacity is analogous to the canonical definition of the capacity associated with a Dirichlet form (see, for example, [2] or [5]). In fact if $\Omega = \mathbf{R}^d$ the two definitions coincide.

As a simple illustration of the capacity of the boundary let $d = 1$, $\Omega = \langle 0, \infty\rangle$ and $H\varphi = -(c\,\varphi')'$ with $c \in W^{1,\infty}_{\rm loc}(\overline{\Omega})$ and $c > 0$. Then there are two possibilities, either $c(0) > 0$, or $c(0) = 0$. In the first case $\operatorname{cap}(\{0\}) > 0$ but in the second $\operatorname{cap}(\{0\}) = 0$. It is important for the latter result that the Lipschitz continuity of c implies that $c(x) \leq a\,x$ for some $a > 0$ and all $x \in [0, 1]$. Note that if $c(x) = O(x^{(1+\delta)})$ as $x \to 0_+$ then the corresponding

Riemannian distance satisfies $d(1\,;x) = \int_x^1 c^{-1/2} = O(x^{(1-\delta)/2})$ as $x \to 0_+$. Thus if $\delta \in [0,1\rangle$ the distance to the boundary, i.e. to the origin, is finite but $c(\{0\}) = 0$.

3. Markov Uniqueness

We begin a more detailed discussion of uniqueness by describing various characterizations of Markov uniqueness. Since the quadratic form associated with each submarkovian extension of H is a Dirichlet form the description of all such extensions is reduced to analyzing all Dirichlet form extensions of the quadratic form h associated with H. Two examples have already occurred, the closure h_D of h and the form h_N used in the definition of the capacity. The form h_N is closed as a direct consequence of the strict ellipticity assumption $C > 0$ (see [13], Section 1, or [11], Proposition 2.1) and both forms are Dirichlet forms by standard estimates. The self-adjoint operator H_N associated with h_N is the submarkovian extension of H corresponding to Neumann boundary conditions and H_D is the extension corresponding to Dirichlet boundary conditions. In general the two submarkovian extensions H_D and H_N of H are distinct. Since $h_N \supseteq h_D$ it follows, however, that one has the ordering $0 \leq H_N \leq H_D$. The significance of the forms h_D and h_N is that they are the minimal and maximal Dirichlet form extensions of h. Versions of this result have been derived under various growth and smoothness assumptions (see [5] Section 3.3.3, [4] Section 3c or [12], Section 2) but the following statement is valid under the general assumptions of Section 1.

Proposition 3.1. *Let k be a Dirichlet form extension of h. Then $h_D \subseteq k \subseteq h_N$. Thus if K is the submarkovian extension of H corresponding to k one has $H_N \leq K \leq H_D$.*

In particular, H is Markov unique if and only if $h_D = h_N$.

Proof. The proof is based on elliptic regularity and some standard results in the theory of Dirichlet forms.

First one clearly has $h_D \subseteq k$. Hence $K \leq H_D$. Secondly, since H is locally strongly elliptic it follows from the usual elliptic regularity arguments that $C_c^\infty(\Omega)D(K) \subseteq D(\overline{H})$ where $\overline{H}$ is the L_2-closure of H (see [12], Corollary 2.3, and [13], Lemma 2.2). Thirdly for each $\chi \in C_c^\infty(\Omega)$ with $0 \leq \chi \leq 1$ define the truncated form k_χ by $D(k_\chi) = D(k) \cap L_\infty(\Omega)$ and $k_\chi(\varphi) = k(\varphi, \chi\varphi) - 2^{-1}k(\chi, \varphi^2)$. Then $k_\chi(\varphi) \leq k(\varphi)$ (see [2], Proposition 4.1.1). Moreover, if $\varphi \in D(K) \cap L_\infty(\Omega)$ then

$$k_\chi(\varphi) = (\varphi, \overline{H}\chi\varphi) - 2^{-1}(H\chi, \varphi^2)\;.$$

But if $\chi_1 \in C_c^\infty(\Omega)$ with $\chi_1 = 1$ on $\operatorname{supp}\chi$ then $\varphi_1 = \chi_1\varphi \in D(\overline{H}) \subseteq W^{2,2}_{\mathrm{loc}}(\Omega)$, where the last inclusion again uses elliptic regularity, and

$$k_\chi(\varphi) = (\varphi_1, \overline{H}\chi\varphi_1) - 2^{-1}(H\chi, \varphi_1^2) = \int_\Omega \chi\,\Gamma(\varphi_1)\;.$$

Combining these observations one has

$$\int_\Omega \chi\,\Gamma(\varphi_1) = k_\chi(\varphi) \leq k(\varphi)$$

for all $\varphi \in D(K) \cap L_\infty(\Omega)$. Then if V is a relatively compact subset of Ω there is a $\mu_V > 0$ such that $C(x) \geq \mu_V I$ for all $x \in V$. Therefore choosing χ such that $\chi = 1$ on V one deduces that $\mu_V \int_V |\nabla\varphi|^2 \leq k(\varphi)$ for each choice of V. Thus $\varphi \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Moreover, $\int_V \Gamma(\varphi) \leq k(\varphi)$ for each V so $\varphi \in D(h_N)$. Consequently $D(K) \cap L_\infty(\Omega) \subseteq D(h_N)$ and

$$h_N(\varphi) = \sup_V \int_V \Gamma(\varphi) \leq k(\varphi)$$

for all $\varphi \in D(K) \cap L_\infty(\Omega)$. But since K is the generator of a submarkovian semigroup $D(K) \cap L_\infty(\Omega)$ is a core of K. In addition $D(K)$ is a core of k. Therefore the last inequality extends by continuity to all $\varphi \in D(k)$. In particular $D(k) \subseteq D(h_N)$. Hence $k \subseteq h_N$ and $H_N \leq K$. □

Remark 3.2. The symmetric operator H also has a largest and a smallest positive self-adjoint extension. The largest extension is the Friedrichs' extension H_D but the smallest extension, the Krein extension, is not generally equal to H_N (see, for example, [5], Theorem 3.3.3).

The identity $h_D = h_N$, in one guise or another, has been the basis of much of the analysis of Markov uniqueness (see, for example, [5], Section 3.3, or [4], Chapter 3). Since h_N is an extension of h_D the identity is equivalent to the condition $D(h_D) = D(h_N)$. But $D(h_D)$ is the closure of $C_c^\infty(\Omega)$ with respect to the graph norm $\varphi \mapsto \|\varphi\|_{D(h_D)} = (h_D(\varphi) + \|\varphi\|_2^2)^{1/2}$. Therefore $h_D = h_N$ if and only if $C_c^\infty(\Omega)$ is a core of h_N. Equivalently, $h_D = h_N$ if and only if $(D(h_D) \cap L_\infty(\Omega))_c$, the space of bounded functions in $D(h_D)$ with compact support in Ω, is a core of h_N.

The next theorem gives two different characterizations of Markov uniqueness which involve the capacity. They are both consequences of the equality $h_D = h_N$.

Theorem 3.3. *Consider the following conditions:*

I. *H is Markov unique,*

II. *for each bounded subset A of $\overline{\Omega}$ there exist $\eta_1, \eta_2, \ldots \in C_c^\infty(\Omega)$ such that*
$\lim_{n\to\infty} \|\mathbb{1}_A \Gamma(\eta_n)\|_1 = 0$ *and* $\lim_{n\to\infty} \|\mathbb{1}_A(\mathbb{1}_\Omega - \eta_n)\varphi\|_2 = 0$ *for each* $\varphi \in L_2(\Omega)$,

III. $\mathrm{cap}(\partial\Omega) = 0$.

Then I⇒II *and* I⇒III. *Moreover, if $B(r)$ is bounded for all $r > 0$ then* II⇒I, III⇒I *and all three conditions are equivalent.*

The second condition of the theorem is a version of capacity estimates introduced by Maz'ya (see [9] Section 2.7) in his analysis of the equality $h_D = h_N$ in the special case $\Omega = \mathbf{R}^d$. In fact his arguments extend to general Ω (see [5] Section 3.3) and give the following result.

Proposition 3.4. *The following conditions are equivalent:*

I. *H is Markov unique,*

II. *for each subset A of $\overline{\Omega}$ with* $\mathrm{cap}(A) < \infty$ *there exist $\eta_1, \eta_2, \ldots \in C_c^\infty(\Omega)$ such that* $\lim_{n\to\infty} \|\mathbb{1}_A \Gamma(\eta_n)\|_1 = 0$ *and* $\lim_{n\to\infty} \|\mathbb{1}_A(\mathbb{1}_\Omega - \eta_n)\varphi\|_2 = 0$ *for each* $\varphi \in L_2(\Omega)$, We will not elaborate on the proof but refer to the relevant sections of [9] and [5]. It should be emphasized that the proposition is valid without any assumption on the Riemannian geometry or the growth of the coefficients of H. Moreover, sets of finite capacity are not necessarily bounded.

Now returning to Theorem 3.3 one sees that the implication I⇒II follows from Proposition 3.4 since bounded subsets automatically have finite capacity. The converse implication II⇒I is not, however, valid without some growth assumption. We briefly describe how it can be established assuming $\rho(x) \to \infty$ as $x \to \infty$, i.e. the Riemannian balls are bounded.

II⇒I It is necessary to prove that $D(h_N) = D(h_D)$. The proof is in two steps. The first step establishes that $(D(h_N) \cap L_\infty(\Omega))_c$, the subspace of $D(h_N)$ spanned by the bounded functions with compact support, is a core of h_N. The argument is quite standard and is based on the assumption that the balls $B(r)$ are bounded. Let τ be a C^∞-function on the right half line satisfying $0 \le \tau \le 1$, $\tau(s) = 1$ if $s \le 1$, $\tau(x) = 0$ if $s \ge 2$ and $|\tau'| \le 2$. Then set $\tau_n = \tau \circ (n^{-1}\rho)$. Since the $B(r)$ are bounded τ_n has compact support. Moreover, $\tau_n(x) \to 1$ as $n \to \infty$ for all $x \in \Omega$. But $\Gamma(\rho) \le 1$. So one also has $\|\Gamma(\tau_n)\|_\infty \le 4\,n^{-2}$. Then fix $\varphi \in D(h_N) \cap L_\infty(\Omega)$ and set $\varphi_n = \tau_n\varphi$. It follows that $\varphi_n \in (D(h_N) \cap L_\infty(\Omega))_c$. But one estimates straightforwardly that φ_n converges in the $D(h_N)$-graph norm to φ. We omit the details.

The second step consists of proving that each $\varphi \in (D(h_N) \cap L_\infty(\Omega))_c$ can be approximated in the $D(h_N)$-graph norm by a sequence $\varphi_n \in D(h_D) \cap L_\infty(\Omega)$. This immediately implies that $h_N = h_D$.

Let $A = \operatorname{supp}\varphi$. If η_n is the sequence in Condition II corresponding to A set $\varphi_n = \eta_n\varphi$. It follows that $\varphi_n \in D(h_N) \cap L_\infty(\Omega)$. But $\operatorname{supp}\varphi_n \subset \Omega$. Hence $\varphi_n \in D(h_D) \cap L_\infty(\Omega)$. Moreover,

$$\lim_{n\to\infty} \|\varphi - \varphi_n\|_2 = \lim_{n\to\infty} \|\mathbf{1}_A(\mathbf{1}_\Omega - \eta_n)\,\varphi\|_2 = 0\;.$$

In addition, since $\nabla(\varphi_n - \varphi) = (\nabla\eta_n)\,\varphi + (1-\eta_n)\,(\nabla\varphi)$, one has

$$\Gamma(\varphi_n - \varphi) \le 2\,\Gamma(\eta_n)\,\varphi^2 + 2\,(1-\eta_n)^2\,\Gamma(\varphi)\;.$$

Then since $\operatorname{supp}\varphi_n \subseteq \operatorname{supp}\varphi = A$ one has

$$h_N(\varphi - \varphi_n) = \|\mathbf{1}_A\Gamma(\varphi_n - \varphi)\|_1 \le 2\,\|\mathbf{1}_A\Gamma(\eta_n)\|_1\,\|\varphi\|_\infty^2 + 2\,\|\mathbf{1}_A(\mathbf{1}_\Omega - \eta_n)\chi\|_2^2$$

where $\chi = \Gamma(\varphi)^{1/2} \in L_2(\Omega)$. Therefore $h_N(\varphi - \varphi_n) \to 0$ as $n \to \infty$.

The equivalence of Conditions I and III was established in [12] and [13] under slightly different growth assumptions. Again we sketch the arguments.

I⇒III Assume $\psi \in D(h_N)$ and $\psi = 1$ on $U \cap \Omega$ where $U \subset \mathbf{R}^d$ is an open neighbourhood of $\partial\Omega$. Since $h_D = h_N$ there is a sequence $\psi_n \in C_c^\infty(\Omega)$ such that $\lim_{n\to\infty} \|\psi - \psi_n\|_{D(h_N)} = 0$. But since $\psi_n \in C_c^\infty(\Omega)$ there are open neighbourhoods U_n of $\partial\Omega$ such that $\psi - \psi_n = 1$ on $U_n \cap \Omega$. Therefore $\operatorname{cap}(\partial\Omega) = 0$.

III⇒I Again since the $B(r)$ are bounded $(D(h_N) \cap L_\infty(\Omega))_c$ is a core of h_N. Hence it suffices to show that each $\varphi \in (D(h_N) \cap L_\infty(\Omega))_c$ can be approximated in the $D(h_N)$-graph norm by a sequence $\varphi_n \in D(h_D)$. Then $h_N = h_D$ and H is Markov unique. But if $A = \operatorname{supp}\varphi \cap \partial\Omega$ then $\operatorname{cap}(A) = 0$ and one can choose $\psi_n \in D(h_N)$ and open sets $U_n \subset \mathbf{R}^d$ such that $A \subset U_n$, $\psi_n = 1$

on $U_n \cap \Omega$ and $\|\psi_n\|_{D(h_N)} \to 0$ as $n \to \infty$. Moreover, since h_N is a Dirichlet form one may also assume $0 \le \psi_n \le 1$. Setting $\varphi_n = (1-\psi_n)\varphi$ it follows that $\varphi_n \in D(h_D) \cap L_\infty(\Omega)$ and

$$\|\varphi - \varphi_n\|_{D(h_N)} = \|\psi_n \varphi\|_{D(h_N)} \le 2\,\|\psi_n\|_{D(h_N)}\|\varphi\|_\infty + 2\int_\Omega \psi_n^2\,\Gamma(\varphi)\;.$$

But both terms on the right converge to zero as $n \to \infty$. The convergence of the second term follows by an equicontinuity estimate since $\|\psi_n\|_2 \to 0$ as $n \to \infty$ and $\Gamma(\varphi) \in L_1(\Omega)$. Therefore φ_n converges to φ in the $D(h_N)$-graph norm.

The most interesting point of Theorem 3.3 is that it provides a practical means of verifying Markov uniqueness. It suffices to establish two properties, $\rho(x) \to \infty$ as $|x| \to \infty$ and $\mathrm{cap}(\partial\Omega) = 0$.

The first of these conditions can be verified by calculating the largest eigenvalue $\lambda(x)$ of $C(x)$, setting $\mu(r) = \sup\{\lambda(x) : x \in \Omega\,,\, |x| < r\}$ and noting that $\rho(x) \ge \nu(|x|)$ where $\nu(r) = \int_0^r \mu^{-1/2}$. Thus if, for example, $\mu(r) \sim r^2(\log r)^\alpha$ as $r \to \infty$ then $\nu(r) \sim (\log r)^{1-\alpha/2}$ as $r \to \infty$. Therefore $\rho(x) \to \infty$ as $x \to \infty$ for all $\alpha \in [0,2\rangle$.

The second condition is a bit more complicated. Its verification is simplified by noting that it suffices to establish that $\mathrm{cap}(B) = 0$ for each bounded subset $B \subseteq \partial\Omega$. But $\mathrm{cap}(B)$ depends on two gross features of H and Ω, the order of degeneracy of the coefficients c_{kl} at the boundary and the dimension of the boundary. First to assign a dimension to B set $B_\delta = \{x \in \Omega : \rho_B(x) < \delta\}$. Then assume there is a $d(B) \in [0,d\rangle$ such that $\sup_{\delta\in\langle 0,1]} \delta^{-(d-d(B))}|B_\delta| < \infty$. Secondly to quantify the degeneracy of the coefficients define $\rho_B(x) = \{d(x\,;y) : y \in \partial\Omega\}$ and assume $C(x) \le a\,\rho_B(x)^{\gamma(B)} I$ for some $a > 0$, $\gamma(B) \ge 0$ and all $x \in B_1$. It then follows by elementary estimation that $\mathrm{cap}(B) = 0$ whenever $\gamma(B) \ge 2 - (d - d(B))$ (see [12] Proposition 4.2). In particular $\mathrm{cap}(B) = 0$ if $\gamma(B) \ge 2$ or $d(B) \le d-2$. Alternatively if B is Lipschitz then $d(B) = d-1$ and it suffices that $\gamma(B) \ge 1$.

4. L_1-UNIQUENESS

The L_1-uniqueness property is in principle stronger than Markov uniqueness and consequently is valid only in more restrictive circumstances. We will describe an analogue of Theorem 3.3 based on two general criteria for L_1-uniqueness. The first has already been alluded to in the introduction. The operator H is L_1-unique if and only if the L_1-closure $\overline{H}^{\,1}$ is the generator of an L_1-continuous semigroup (see [4], Theorem 1.2 in Appendix 1.A). Alternatively this criterion can be expressed in terms of the submarkovian semigroup S generated by the Friedrichs' extension H_D of H on $L_2(\Omega)$. The semigroup S extends from $L_2(\Omega) \cap L_p(\Omega)$ to a contraction semigroup on $S^{(p)}$ for all $p \in [1,\infty]$ which is strongly continuous if $p \in [1,\infty\rangle$ and weakly* continuous if $p = \infty$. If H_p denotes the generator of $S^{(p)}$ on $L_p(\Omega)$ then H is L_1-unique if and only if $H_1 = \overline{H}^{\,1}$.

The second criterion is the conservation of probability for the semigroup S. This is expressed in terms of $S^{(1)}$ acting on $L_1(\Omega)$. The conservation criterion has been derived in several settings (see [7] [1] [3] [6] and references therein). It is equivalent to the semigroup being Markovian, a property expressed in terms of the dual semigroup $S^{(\infty)}$ acting on $L_\infty(\Omega)$.

Proposition 4.1. *The following conditions are equivalent:*

I. *H is L_1-unique,*

II. *S conserves probability, i.e. $\|S^{(1)}_t\varphi\|_1 = \|\varphi\|_1$ for all positive $\varphi \in L_1(\Omega)$ and $t > 0$,*

III. *S is Markovian, i.e. $S^{(\infty)}_t \mathbb{1}_\Omega = \mathbb{1}_\Omega$ for all $t > 0$.*

Proof. The equivalence II$\Leftrightarrow$III follows from duality since $(S^{(1)}_t)^* = S^{(\infty)}_t$ for all $t > 0$.

The implication I$\Rightarrow$II is straightforward. Clearly $(H\chi, \mathbb{1}_\Omega) = 0$ for all $\chi \in C_c^\infty(\Omega)$. But L_1-uniqueness is equivalent to the condition $H_1 = \overline{H}^{\,1}$. Therefore $(H_1\chi, \mathbb{1}_\Omega) = 0$ for all $\chi \in D(H_1)$ by closure. Then

$$\|S^{(1)}_t\varphi\|_1 - \|\varphi\|_1 = (S^{(1)}_t\varphi, \mathbb{1}_\Omega) - (\varphi, \mathbb{1}_\Omega) = \int_0^t ds\, (H_1 S^{(1)}_s\varphi, \mathbb{1}_\Omega) = 0$$

for all positive $\varphi \in D(H_1)$. Then Condition II follows by density.

The real content of the proposition is contained in the implication III$\Rightarrow$I. The proof of which is a consequence of the following lemma (see [1], Corollary 2.5 or [3], Lemma 2.3).

Lemma 4.2. *If $\psi \in L_\infty(\Omega)$, $\psi \geq 0$ and $\xi = (I + H_\infty)^{-1}\psi$ then*

$$((I + H)\chi, \xi) = (\chi, \psi)$$

for all $\chi \in C_c^\infty(\Omega)$.

Moreover, if $\eta \geq 0$ is a continuous function with

$$((I + H)\chi, \eta) \geq (\chi, \psi)$$

for all positive $\chi \in C_c^\infty(\Omega)$ then $\eta \geq \xi$.

We refer to [3] for the proof but emphasize that it is independent of the growth of the coefficients c_{kl} at infinity. The proof only involves the Friedrichs' extensions of H and its restrictions $H|_{C_c^\infty(V)}$ to relatively compact subsets V of Ω.

The proof of the implication III$\Rightarrow$I in Proposition 4.1 now proceeds by negation.

First since H is L_1-dissipative its closure is the generator of a strongly continuous semigroup if and only if the range of $I + H$ is L_1-dense. Assume the contrary, i.e. assume there is a $\varphi \in L_\infty(\Omega)$ with $\|\varphi\|_\infty = 1$ such that

$$((I + H)\chi, \varphi) = 0$$

for all $\chi \in C_c^\infty(\Omega)$. Then $\varphi \in D(H_V^*)$ with $H_V = H|_{C_c^\infty(V)}$ for each relatively compact subset V of Ω where the star indicates the L_2-adjoint. Since H_V is strongly elliptic it follows by elliptic regularity that φ is continuous. Set $\eta = \mathbb{1}_\Omega - \varphi$. Then $\eta \geq 0$, η is continuous and

$$((I + H)\chi, \eta) = ((I + H)\chi, \mathbb{1}_\Omega) - ((I + H)\chi, \varphi) = (\chi, \mathbb{1}_\Omega)$$

for all $\chi \in C_c^\infty(\Omega)$ since $(H\chi, \mathbb{1}_\Omega) = 0$. Therefore

$$\eta \geq (I + H_\infty)^{-1}\mathbb{1}_\Omega$$

by Lemma 4.2, i.e. $\mathbb{1}_\Omega - (I + H_\infty)^{-1}\mathbb{1}_\Omega = \varphi \neq 0$. Hence S is not Markovian. □

The characterization of L_1-uniqueness in terms of conservation of probability then allows the extension of Theorem 3.3.

Theorem 4.3. *Consider the following conditions:*

I. *H is L_1-unique,*
II. *H is Markov unique,*
III. *for each bounded subset A of $\overline{\Omega}$ there exists a sequence $\eta_1, \eta_2, \ldots \in C_c^\infty(\Omega)$ such that $\lim_{n\to\infty} \|\mathbb{1}_A \Gamma(\eta_n)\|_1 = 0$ and $\lim_{n\to\infty} \|\mathbb{1}_A(\mathbb{1}_\Omega - \eta_n)\,\varphi\|_2 = 0$ for each $\varphi \in L_2(\Omega)$,*
IV. $\operatorname{cap}(\partial\Omega) = 0$.

Then I⇒II⇒III *and* II⇒IV*. Moreover, if $B(r)$ is bounded for all $r > 0$ then* I⇒II⇔III⇔IV*. If, in addition, there are $a, b > 0$ such that $|B(r)| \leq a\, e^{b\, r^2}$ for all $r > 0$ then* IV⇒I *and all four conditions are equivalent.*

The implication I⇒II is evident and the implications II⇒III and II⇒IV follow directly from Theorem 3.3. Moreover, if the balls $B(r)$ are bounded then the earlier theorem also establishes that III⇒II and IV⇒II. Although the boundedness of the balls suffices to establish Markov uniqueness from the capacity estimates it is not sufficient to establish L_1-uniqueness. Example 4.2 in [13] gives a one-dimensional example of an H with a coefficient $c(|x|) \sim x^2(\log|x|)(\log(\log|x|))$ as $x \to \infty$ which is Markov unique but not L_1-unique. In this example the balls $B(r)$ are finite intervals and the growth bounds $|B(r)| \leq a\, e^{b\, r^2}$ fail. But the bounds only just fail. For each $\varepsilon > 0$ there are $a, b > 0$ such that $|B(r)| \leq a\, e^{b\, r^{2+\varepsilon}}$ for all $r > 0$. If, however, $c(|x|) \sim x^2(\log|x|)$ then the one-dimensional operator is L_1-unique because the bounds $|B(r)| \leq a\, e^{b\, r^2}$ are valid.

The only current proof of IV⇒I is based on the ideas of [12] [13] and is quite lengthy. Nevertheless, the strategy of the proof is straightforward. It begins by establishing that if Ω is bounded, or more generally if $|\Omega| < \infty$, then II⇒I. This is straightforward since one then has $L_\infty(\Omega) \subseteq L_2(\Omega)$ and so $\mathbb{1}_\Omega \in L_2(\Omega)$. Moreover, $\mathbb{1}_\Omega \in D(H_N)$, $H_N\mathbb{1}_\Omega = 0$ and the semigroup T generated by H_N is Markovian. But $S = T$ by Markov uniqueness. Thus S is Markovian and H is L_1-unique by Proposition 4.1. Then the idea is to exploit the result through an approximation of H by a sequence of operators H_n acting on a family of increasing subspaces $L_2(\Omega_n)$ where the Ω_n are bounded. If τ_n is a pointwise increasing sequence of C^∞-functions with $0 \leq \tau_n \leq 1$, $\tau_n = 1$ if $|x| \leq n/2$ and $\tau_n = 0$ if $|x| \geq n$ then one defines H_n as the operator with coefficients $\tau_n c_{kl}$ acting on $\Omega_n = \Omega \cap \{x : |x| < n\}$. It follows from the assumption $\operatorname{cap}(\partial\Omega) = 0$ that $\operatorname{cap}_n(\partial\Omega_n) = 0$ where cap_n is the capacity relative to H_n on $L_2(\Omega_n)$. Then the H_n are Markov unique. But the corresponding Markovian semigroups $T^{(n)}$ on $L_2(\Omega_n)$ can be extended to Markovian semigroups $\widehat{S}^{(n)}$ on $L_2(\Omega)$ by setting $\widehat{S}^{(n)}_t = T^{(n)}_t \oplus \mathbb{1}_{\Omega'_n}$ on $L_2(\Omega)$ where $\Omega'_n = \Omega \cap \{x : |x| > n\}$. Then using monotonicity arguments one establishes that the Markovian semigroups $\widehat{S}^{(n)}$ are L_2-convergent to the semigroup S generated by the Friedrichs' extension of H. But L_2-convergence does not imply that S is Markovian, for this one requires L_1-convergence.

The latter can, however, be deduced from combining the L_2-convergence with Davies–Gaffney off-diagonal Gaussian bounds. It is in this latter part of the proof that the Gaussian growth bounds on the balls $B(r)$ are needed. The form of the bound is a reflection of the Gaussian decrease of the Davies–Gaffney bounds.

The proof of equivalence of L_1-uniqueness and Markov uniqueness is rather lengthy because it allows a very general volume growth. If the assumption of Gaussian growth, $|B(r)| \leq a\, e^{b\, r^2}$ as $r \to \infty$, is somewhat relaxed then there is an alternative, simpler, proof which we next describe. This proof covers exponential volume growth which arises if the norm $\|C(x)\|$ of the matrix of coefficients $C(x)$ satisfies bounds $\|C(x)\| \leq \lambda\,(1 + x^2)$ for all $x \in \Omega$.

Theorem 4.4. *Assume the balls $B(r)$ are bounded for all $r > 0$ and there are $a, b > 0$ such that $|B(r)| \leq a\, e^{b\, r}$ for all $r > 0$. Then the following conditions are equivalent:*

I. *H is L_1-unique,*
II. *H is Markov unique.*

Proof. Clearly I⇒II and it suffices to prove the converse. Therefore assume H is Markov unique. But H is L_1-unique if and only if its L_1-closure is the generator of a continuous semigroup and since H is L_1-dissipative this is the case if and only if $(\lambda I + H)C_c^\infty(\Omega)$ is dense in $L_1(\Omega)$ for all large positive λ. Now we argue by contradiction.

Suppose there is a non-zero $\psi \in L_\infty(\Omega)$ and a $\lambda > 0$ such that $(\psi, (\lambda I + H)\varphi) = 0$ for all $\varphi \in C_c^\infty(\Omega)$. Then for each relatively compact open subset U of Ω one has $\psi \in D(H_U^*)$ where H_U^* denotes the $L_2(U)$-adjoint of the restriction $H_U = H|_{C_c^\infty(U)}$. But H_U is strongly elliptic on $L_2(U)$ and it follows from elliptic regularity that $\psi \in W^{2,2}_{\rm loc}(U)$. In particular $\eta\,\psi, \eta^2\psi \in W^{2,2}(U)$ for all $\eta \in C_c^\infty(U)$. Then $(\psi, (\lambda I + \overline{H}_U)\varphi) = 0$ for all $\varphi \in W^{2,2}(U)$ where $\overline{H}_U$ denotes the L_2-closure of H_U. Thus setting $\varphi = \eta^2\psi$ one has

$$\lambda\,\|\eta\,\psi\|_2^2 = -(\psi, \overline{H}_U\,\eta^2\psi) = (\psi, \Gamma(\eta)\psi) - (\eta\,\psi, \overline{H}_U\,\eta\,\psi)\;.$$

It follows by positivity of H_U that

$$\lambda\,\|\eta\,\psi\|_2^2 \leq (\psi, \Gamma(\eta)\psi) \tag{5}$$

for all $\eta \in C_c^\infty(U)$ and for all relatively compact open subsets U of Ω. Thus (5) is valid for all $\eta \in C_c^\infty(\Omega)$. But since $\psi \in L_\infty(\Omega)$ one has $\|\eta\,\psi\|_2 \leq \|\eta\|_2\|\psi\|_\infty$. In addition

$$|(\psi, \Gamma(\eta)\psi)| \leq \|\Gamma(\eta)\|_1\|\psi\|_\infty^2 = h_D(\eta)\,\|\psi\|_\infty^2\;.$$

Hence (5) extends by continuity to all $\eta \in D(h_D)$. Therefore, since $D(h_D) = D(h_N)$ by Markov uniqueness, (5) is valid for all $\eta \in D(h_N)$.

Now for each $n \in \mathbf{N}$ choose a $\tau_n \in C_c^\infty(0, \infty)$ with $0 \leq \tau_n \leq 1$, $\tau_n(y) = 1$ if $0 \leq y \leq n$, $\tau_n(y) = 0$ if $y \geq n + 1$ and $|\nabla\tau_n| \leq \lambda_0$. Then set $\eta_n = \tau_n \circ \rho$. It follows that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ on $B(n)$ and $\eta_n = 0$ on $B(n+1)^{\rm c}$. Since $\Gamma(\rho) \leq 1$ it also follows that $\Gamma(\eta_n) \leq \lambda_0^2\,\eta_{n+1}^2$ on Ω. Therefore $\Gamma(\eta_n) + \eta_n^2 \in L_1(\Omega)$ and $\eta_n \in D(h_N)$. Next set $b_n = e^{-b(n+1)}\|\eta_n\psi\|_2^2$. Then $b_n \leq e^{-b(n+1)}\,|B(n+1)|\,\|\psi\|_\infty^2 \leq a\,\|\psi\|_\infty^2$ uniformly for all $n \in \mathbf{N}$ by the exponential growth bound on the balls $B(r)$. Replacing η in (5) by η_n one deduces that

$$b_n \le e^{-b(n+1)}\,\lambda^{-1}\,(\psi, \Gamma(\eta_n)\psi)$$
$$\le \lambda_0^2\, e^{-b(n+1)}\,\lambda^{-1}\,\|\eta_{n+1}\psi\|_2^2 = (\lambda_0^2\, e^b/\lambda)\, b_{n+1}$$

and, by iteration,

$$b_n \le (\lambda_0^2\, e^b/\lambda)^m\, b_{n+m} \le a\,(\lambda_0^2\, e^b/\lambda)^m\,\|\psi\|_\infty^2$$

for all $m \in \mathbf{N}$. Thus if $\lambda > \lambda_0^2\, e^b$ one concludes in the limit $m \to \infty$ that $b_n = 0$. In particular the $L_2(B(n))$-norm of ψ is zero. Since this conclusion is valid for all n one must have $\psi = 0$ which is a contradiction. So H is L_1-unique. □

The foregoing reasoning is closely related to arguments used by many authors to establish self-adjointness properties of elliptic operators (see [3], Section 3, and references therein). It can also be extended to derive uniqueness of non-symmetric elliptic operators from uniqueness of the symmetric principal part, but that is another story.

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