## 14. RESOLVENT OPERATOR APPROXIMATION

As promised at the end of the last section, we introduce a mode of approximation which includes both the norm approximation and the collectively compact approximation, and which has nice implications for spectral approximation. At the end of this section we also show that the conditions needed for the convergence of various iteration schemes given in Section 11 are fulfilled under this mode of approximation. A variant of this mode has been called 'strong approximation' in the literature (cf. [CL], [LN]), but we have chosen to give it another more appropriate name.

A sequence $(T_n)$ in $BL(X)$ is said to be a <u>resolvent operator approximation of</u> $T \in BL(X)$ if $T_n \xrightarrow{p} T$, and for every $z \in \rho(T)$,

$$\|(T-T_n)R(z)(T-T_n)\| \to 0 . \tag{14.1}$$

We denote this fact by $T_n \xrightarrow{ro} T$. Showing that a sequence $(T_n)$ is a resolvent operator approximation of $T$ is, in general, a formidable task. However, there are two well known modes of approximation which imply the resolvent operator approximation. It is obvious that $T_n \xrightarrow{\|\ \|} T$ implies $T_n \xrightarrow{ro} T$. Also, it follows by letting $A = TR(z)$, $A_n = T_nR(z)$ for $z \in \rho(T)$, and $B = T$, $B_n = T_n$ in Proposition 13.3 that $T_n \xrightarrow{cc} T$ implies $T_n \xrightarrow{ro} T$.

Let $T_n \xrightarrow{p} T$. By the uniform boundedness principle ([L], 9.1 and 9.3),

$$\|T\| \leq \sup\{\|T_n\| : n = 1,2,\ldots\} < \infty .$$

For a closed subset $E$ of $\rho(T)$, we have

$$\max_{z\in E} \|R(z)\| < \infty , \quad \nu_n(E) \equiv \max_{z\in E} \|(T-T_n)R(z)\| < \infty ,$$

$$\nu(E) \equiv \sup_{n=1,2,\ldots} \nu_n(E) < \infty , \tag{14.2}$$

since the function $z \mapsto R(z)$ is continuous on $E$ and $\|R(z)\| \to 0$ as $|z| \to \infty$ by (5.9) .

Considering the analytic functions $f_n(z) = (T-T_n)R(z)(T-T_n)$ on $\rho(T)$ , it follows by Problem 4.3 that the condition (14.1) holds for every $z \in \rho(T)$ if and only if every connected component of $\rho(T)$ contains a set $\{z_k\}$ with a limit point in that component such that the condition (14.1) holds for $z = z_k$ , $k = 1,2,\ldots$ ; in that case, the convergence is uniform for $z$ in any closed subset of a connected component of $\rho(T)$ .

The simplest situation arises when $\rho(T)$ is itself connected. This is certainly the case if $\sigma(T)$ is a countable set, e.g., when $T$ is a compact operator. In this situation, it can be shown that the condition (14.1) holds for every $z \in \rho(T)$ if and only if for each fixed $k = 0,1,2,\ldots$ ,

$$\|(T-T_n)T^k(T-T_n)\| \to 0 \quad \text{as} \quad n \to \infty .$$

See Problem 14.1. If $T_n \xrightarrow{p} T$ and $T$ is compact, then this condition is equivalent to $\|(T-T_n)T_n\| \to 0$ as $n \to \infty$ .

*We assume throughout this section that* $T_n \xrightarrow{ro} T$ .

**PROPOSITION 14.1** Let $E$ be a closed subset of $\rho(T)$ . Then $(T-T_n)R(z) \xrightarrow{p} 0$ , uniformly for $z \in E$ , and

$$\max_{z \in E} \|(T-T_n)R(z)(T-T_n)\| \to 0 , \tag{14.3}$$

$$\delta_n(E) \equiv \max_{z \in E} \|[(T-T_n)R(z)]^2\| \to 0 . \tag{14.4}$$

**Proof** We assume without loss of generality that the set $E$ is compact. Let $\epsilon > 0$ , and find $z_1,\ldots,z_k$ in $E$ such that for every $z \in E$ there is some $z_j$ with $|z-z_j| < \epsilon$ . For $z \in \rho(T)$ , the first resolvent identity (5.5) shows that

$$(T-T_n)R(z) = (T-T_n)[R(z)-R(z_j)] + (T-T_n)R(z_j)$$
$$= (z-z_j)(T-T_n)R(z_j)R(z) + (T-T_n)R(z_j) .$$

Let $x \in X$ . Since $T_n \xrightarrow{p} T$ , there exists $n_0$ such that for all $n \geq n_0$ and $j = 1,\ldots,k$ , we have $\|(T-T_n)R(z_j)x\| < \epsilon$ . Then

$$\|(T-T_n)R(z)x\| \leq \epsilon\left[\nu(E) \max_{z\in E}\|R(z)\| + 1\right] .$$

Hence $(T-T_n)R(z)x \to 0$ , uniformly for $z \in E$ . For $z \in E$ , we have

$$(T-T_n)R(z)(T-T_n) = (z-z_j)(T-T_n)R(z)R(z_j)(T-T_n) + (T-T_n)R(z_j)(T-T_n) .$$

Since for $j = 1,\ldots,k$ , $\|(T-T_n)R(z_j)(T-T_n)\| \to 0$ by assumption, we see that (14.3) holds. Finally, since

$$\delta_n(E) \leq \max_{z\in E} \|(T-T_n)R(z)(T-T_n)\| \max_{z\in E} \|R(z)\|,$$

we see that $\delta_n(E) \to 0$ as $n \to \infty$ . //

*For a closed subset* $E$ *of* $\rho(T)$ , *let* $n_0(E)$ *denote the smallest positive integer such that for all* $n \geq n_0(E)$ , *we have* $\delta_n(E) < 1$ . Such an integer exists by Proposition 14.1.

**COROLLARY 14.2** Let $E \subset \rho(T)$ be closed. Then for all $n \geq n_0(E)$ , we have $E \subset \rho(T_n)$ ; if we let $R_n(z) = (T_n-zI)^{-1}$ for $z \in E$ , then

$$R_n(z) = R(z) \sum_{k=0}^{\infty} [(T-T_n)R(z)]^k , \tag{14.5}$$

$$\max_{z\in E} \|R_n(z)\| \leq \frac{\max_{z\in E} \|R(z)\| \; [1 + \nu_n(E)]}{1 - \delta_n(E)} ,$$

so that

$$M(E) \equiv \sup\{\|R_n(z)\| : z \in E , n \geq n_0(E)\} < \infty , \tag{14.6}$$

$$\max_{z\in E} \|R_n(z)x - R(z)x\| \to 0 \quad \text{for every} \quad x \in X . \tag{14.7}$$

**Proof** Let $n \geq n_0(E)$ . Then

$$\max_{z \in E} r_\sigma((T-T_n)R(z)) \leq \max_{z \in E} \|[(T-T_n)R(z)]^2\|^{1/2} = \sqrt{\delta_n(E)} < 1 .$$

Let $z \in E$ , $A = T - zI$ and $B = T_n - zI$ . By Theorem 9.1, $B$ is invertible, i.e., $z \in \rho(T_n)$ . Thus, $E \subset \rho(T_n)$ . Also, (9.4) shows that $R_n(z)$ is given by (14.5), and

$$\|R_n(z)\| \leq \frac{\|R(z)\| \ \|I + (T-T_n)R(z)\|}{1 - \|[(T-T_n)R(z)]^2\|} .$$

by (9.5). Taking the maximum over $z \in E$ , we obtain (14.6). By (14.2) and (14.4), $\max_{z \in E} \|R_n(z)\|$ , $n \geq n_0(E)$ , remains bounded.
Finally, let $x \in X$ . Then for $z \in E$ and $n \geq n_0(E)$ ,

$$R_n(z)x - R(z)x = R_n(z)(T-T_n)R(z)x$$

by the second resolvent identity (9.2). Hence

$$\|R_n(z)x - R(z)x\| \leq M(E)\left[\max_{z \in E} \|(T-T_n)R(z)x\|\right] .$$

But $\max_{z \in E} \|(T-T_n)R(z)x\| \to 0$ by Proposition 14.1. Hence (14.7) follows. //

We now prove the upper semicontinuity of the spectrum with respect to the resolvent operator approximation.

**THEOREM 14.3** Let $T_n \xrightarrow{ro} T$ . Let $G$ be an open set containing $\sigma(T)$ . Then $G$ also contains $\sigma(T_n)$ for all large $n$ .

If $\lambda_n \in \sigma(T_n)$ and $\lambda_n \to \lambda$ , then $\lambda \in \sigma(T)$ .

**Proof** The set $E = \{z \in \mathbb{C} : z \notin G , |z| \leq \alpha\}$ , where $\alpha = \sup\{\|T_n\| : n = 1,2,\dots\}$ , is a compact subset of $\rho(T)$ . Hence by Corollary 14.2, we see that $E \subset \rho(T_n)$ for all $n \geq n_0(E)$ ; but if

$z \in \sigma(T_n)$ , then $|z| \le r_\sigma(T_n) \le \|T_n\| \le \alpha$ . This shows that $\sigma(T_n) \subset G$ for all $n \ge n_0(E)$ .

Let $\lambda_n \in \sigma(T_n)$ and $\lambda_n \to \lambda$ . Assume $\lambda \in \rho(T)$ , and let $d = \mathrm{dist}(\lambda,\sigma(T))$ . Then the set $G = \{z \in \mathbb{C} : \mathrm{dist}(z,\sigma(T)) < d/2\}$ is open and contains $\sigma(T)$ . Hence $\sigma(T_n) \subset G$ for all large $n$ , so that $\mathrm{dist}(\lambda_n,\sigma(T)) < d/2$ . Now,

$$\begin{aligned} d = \mathrm{dist}(\lambda,\sigma(T)) &\le |\lambda-\lambda_n| + \mathrm{dist}(\lambda_n,\sigma(T)) \\ &< |\lambda-\lambda_n| + d/2 . \end{aligned}$$

This shows that $|\lambda-\lambda_n| > d/2$ for all large $n$ , and contradicts $\lambda_n \to \lambda$ . Thus, $\lambda \in \sigma(T)$ . //

Before we proceed to prove the lower semicontinuity of the spectrum at the discrete spectral values, we prove a useful preliminary result.

**LEMMA 14.4** Let $\Gamma$ be a simple closed rectifiable positively oriented curve in $\rho(T)$ . Then $\Gamma \subset \rho(T_n)$ for all $n \ge n_0(\Gamma)$. If $P$ and $P_n$ denote the spectral projections of $T$ and $T_n$ associated with $\Gamma$ respectively, then

$$\begin{aligned} &P_n \xrightarrow{p} P , \\ (14.8) \qquad &\mathrm{rank}\, P_n = \mathrm{rank}\, P , \quad n \ge n_0(\Gamma) . \end{aligned}$$

If $\mathrm{rank}\, P < \infty$ , then

$$P_n \xrightarrow{cc} P ,$$

$$(14.9) \qquad \|(T-T_n)P\| \to 0 , \ \|(T-T_n)P_n\| \to 0 ,$$

$$(14.10) \qquad \|(P-P_n)P\| \to 0 , \ \|(P-P_n)P_n\| \to 0 ,$$

$$(14.11) \qquad \hat{\delta}(P(X),P_n(X)) \to 0 .$$

**Proof** Since $\Gamma$ is a compact subset of $\rho(T)$ , it follows by Corollary 14.2 that $\Gamma \subset \rho(T_n)$ for all $n \ge n_0(\Gamma)$ . Fix $n \ge n_0(\Gamma)$ . Consider the family of operators

$$T_n(t) = T + t(T_n - T) \ , \ t \in \mathbb{C} \ ,$$

and the disk

$$\partial_\Gamma = \left\{ t \in \mathbb{C} : |t| < \frac{1}{\max_{z\in\Gamma} r_\sigma((T_n-T)R(z))} \right\} ,$$

as introduced in (9.14). Since $\delta_n(\Gamma) < 1$, we see that $t \in \partial_\Gamma$ for all $|t| \leq 1$. If $P_n(t)$ denotes the spectral projection associated with $T_n(t)$ and $\Gamma$, then by Corollary 9.7 we have for $t \in \partial_\Gamma$

$$\text{rank } P_n = \text{rank } P_n(1) = \text{rank } P_n(0) = \text{rank } P \ .$$

Next, by Proposition 14.1, we see that for each $x \in X$, $R_n(z)x \to R(z)x$ uniformly for $z \in \Gamma$. Hence by (4.8),

$$P_n x = -\frac{1}{2\pi i}\int_\Gamma R_n(z)x \, dz \to -\frac{1}{2\pi i}\int_\Gamma R(z)x \, dz = Px \ ,$$

i.e., $P_n \xrightarrow{p} P$.

Now, assume that rank $P < \infty$. Then by Theorem 13.4, we have $P_n \xrightarrow{cc} P$. The relations (14.9) and (14.10) follow immediately from (13.5).

Since the gap between the subspaces $P(X)$ and $P_n(X)$ satisfies (cf. (2.4))

$$\hat{\delta}(P(X), P_n(X)) \leq \max \{\|(P-P_n)P\| \ , \ \|(P_n-P)P_n\|\} \ ,$$

it follows from (14.10) that, it tends to zero as $n \to \infty$. //

Now we prove a very important theorem.

**THEOREM 14.5** Let $\lambda$ be a discrete spectral value of $T$ of algebraic multiplicity $m$, geometric multiplicity $g$, and let it be a pole of order $\ell$ of $R(z)$ . Let $\Gamma$ be a curve in $\rho(T)$ separating $\lambda$ from the rest of $\sigma(T)$ .

(a) For each $n \geq n_0(\Gamma)$ , $\sigma(T_n) \cap \operatorname{Int} \Gamma$ consists of a finite number of eigenvalues $\lambda_{n,1},\ldots,\lambda_{n,k_n}$ of $T_n$ , where $k_n \leq m$ .

(b) If $\lambda_n \in \{\lambda_{n,1},\ldots,\lambda_{n,k_n}\}$ , then as $n \to \infty$ ,

$$\lambda_n \to \lambda , \tag{14.12}$$

and if $\varphi_n$ is an eigenvector of $T_n$ corresponding to $\lambda_n$ of norm 1 , then

$$\|\varphi_n - P\varphi_n\| \leq \frac{\ell(\Gamma)}{\operatorname{dist}(\lambda,\Gamma)} \max_{z\in\Gamma} \|R(z)\| \ \|(T-T_n)\varphi_n\| \to 0 , \tag{14.13}$$

where $\ell(\Gamma)$ is the length of $\Gamma$ . Also, a *subsequence* of $(\varphi_n)$ converges to an eigenvector of $T$ .

(c) Let $m_{n,j}$ (resp., $g_{n,j}$) denote the algebraic (resp., geometric) multiplicity of $\lambda_{n,j}$ , and let $\ell_{n,j}$ be the order of the pole of $R_n(z)$ at $\lambda_{n,j}$ . Then

$$\begin{aligned} m_{n,1} + \ldots + m_{n,k_n} = m \quad &\text{for all} \quad n \geq n_0(\Gamma) , \\ g_{n,j} \leq g \quad \text{for} \quad j = 1,\ldots,k_n \quad &\text{and all large} \quad n , \\ \ell \leq \ell_{n,1} + \ldots + \ell_{n,k_n} \quad &\text{for all large} \quad n . \end{aligned} \tag{14.14}$$

**Proof** (a) For $n \geq n_0(\Gamma)$ , we have $\Gamma \subset \rho(T_n)$ by Lemma 14.4, and if $P$ (resp., $P_n$) denotes the spectral projection associated with $T$(resp., $T_n$) and $\Gamma$ , then

$$\operatorname{rank} P_n = \operatorname{rank} P = m .$$

Hence by Theorem 7.7, $\sigma(T_n) \cap \operatorname{Int} \Gamma$ consists of a finite number of eigenvalues $\lambda_{n,1},\ldots,\lambda_{n,k_n}$ of $T_n$ and we have

$$m_{n,1} + \ldots + m_{n,k_n} = \text{rank } P_n = m .$$

(b) For $\epsilon > 0$, let $\Gamma_\epsilon$ denote the circle with centre $\lambda$ and radius $\epsilon$. Then for all sufficiently small $\epsilon$ and all $n \geq n_0(\Gamma_\epsilon)$, we have by (14.8)

$$\text{rank } P_{\Gamma_\epsilon}(T_n) = \text{rank } P_{\Gamma_\epsilon}(T) = \text{rank } P = m .$$

This implies that for all $n \geq n_0(\Gamma_\epsilon)$, $\lambda_{n,j} \in \text{Int } \Gamma_\epsilon$, i.e., $|\lambda_{n,j}-\lambda| < \epsilon$ for $j = 1,\ldots,k_n$. Thus, if $\lambda_n \in \{\lambda_{n,1},\ldots,\lambda_{n,k_n}\}$, then $\lambda_n \to \lambda$, as $n \to \infty$. Next, let $\varphi_n$ be an eigenvector of $T_n$ corresponding to $\lambda_n$ of norm 1. By the second resolvent identity (5.5), we have

$$\begin{aligned} P_n - P &= \frac{1}{2\pi i}\int_\Gamma [R(z)-R_n(z)]dz \\ &= \frac{1}{2\pi i}\int_\Gamma R(z)(T_n-T)R_n(z)dz . \end{aligned}$$

But $R_n(z)\varphi_n = \varphi_n/(\lambda_n-z)$ for $z \in \rho(T_n)$. Hence

$$\varphi_n - P\varphi_n = (P_n-P)\varphi_n = \left[\frac{1}{2\pi i}\int_\Gamma \frac{R(z)}{\lambda_n - z}\,dz\right](T_n-T)\varphi_n .$$

Since $\lambda_n \to \lambda$, we see that for all $z \in \Gamma$ and for all large $n$

$$|\lambda_n-z| \geq d/2\pi , \quad \text{where} \quad d = \text{dist}(\lambda,\Gamma) .$$

Also,

$$\|(T-T_n)\varphi_n\| = \|(T-T_n)P_n\varphi_n\| \leq \|(T-T_n)P_n\|$$

which converges to zero by (14.9). Hence (14.13) follows.

Since $P$ is compact and $P_n \xrightarrow{cc} P$ by Lemma 14.4, we conclude that the set

$$\cup \{P_n x : \|x\| \le 1 , n \ge n_0(\Gamma)\}$$

is totally bounded in $X$. Now, each $\varphi_n = P_n\varphi_n$ is in this set, so that there is a subsequence $(\varphi_{n_k})$ which converges to some $\varphi$ in $X$. Since $\|(T-T_{n_k})\varphi_{n_k}\| \le \|(T-T_{n_k})P_{n_k}\| \to 0$ by (14.9), we have

$$T\varphi = \lim T\varphi_{n_k} = \lim T_{n_k}\varphi_{n_k} = \lim \lambda_{n_k}\varphi_{n_k} = \lambda\varphi ,$$

i.e., $\varphi$ is an eigenvector of $T$ corresponding to $\lambda$.

(c) We have already shown that $m_{n,1} + \ldots + m_{n,k_n} = m$ for all $n \ge n_0(\Gamma)$. To show $g_{n,j} \le g$ for all large $n$ and $j = 1,\ldots,k_n$, assume that $\dim Z(T_n-\lambda_{n,j_n} I) = g_{n,j_n} \ge h$ for all $n$ in an infinite subset $N$ and $j_n \in \{1,\ldots,k_n\}$. Let $\lambda_n = \lambda_{n,j_n}$. By the Riesz lemma ([L],6.8; cf. Problem 3.1.), there is $x_{n,k} \in Z(T_n-\lambda_n I)$ for $k = 1,\ldots,h$ such that $\|x_{n,k}\| = 1$ and

$$\|x_{n,k} - \sum_{i=1}^{k-1} c_i x_{n,i}\| \ge 1/2 \quad \text{for } k = 2,\ldots,h \text{ and all } c_i \in \mathbb{C} .$$

Since for each $k = 1,\ldots,h$, a subsequence of $(x_{n,k})$ converges to some $x_k \in Z(T-\lambda I)$, we see that $\|x_k\| = 1$ for $k = 1,\ldots,h$ and

$$\|x_k - \sum_{i=1}^{k-1} c_i x_i\| \ge 1/2 \quad \text{for } k = 2,\ldots,h \text{ and all } c_i \in \mathbb{C} .$$

This shows that the set $\{x_1,\ldots,x_h\}$ is linearly independent in $Z(T-\lambda I)$, i.e., $g \ge h$. Thus, $g_{n,j} \le g$ for all large $n$ and $j = 1,\ldots,k_n$.

Finally, by Lemma 7.8 and Lemma 7.1(ii), we have

$$P_n(X) = Z\left[(T_n-\lambda_{n,1})^{\ell_{n,1}}\right] \oplus \ldots \oplus Z\left[(T_n-\lambda_{n,k_n})^{\ell_{n,k_n}}\right] .$$

Let

$$\tilde{T}_n = (T_n - \lambda_{n,1} I)^{\ell_{n,1}} \cdots (T_n - \lambda_{n,k_n} I)^{\ell_{n,k_n}} , \quad n \geq n_0(\Gamma) .$$

Assume that $\ell_{n,1} + \ldots + \ell_{n,k_n} = h$ for all $n$ in an infinite set $N$. Since $T_n \xrightarrow{p} T$, $\lambda_{n,j} \to \lambda$, and $P_n \xrightarrow{cc} P$, we have

$$\tilde{T}_n \xrightarrow{p} (T-\lambda I)^h , \quad \tilde{T}_n P_n \xrightarrow{p} (T-\lambda I)^h P ,$$

as $n$ runs through $N$. But clearly, $\tilde{T}_n P_n = 0$, so that $(T-\lambda I)^h P = 0$, i.e., $R(P) \subset Z((T-\lambda I)^h)$. Since $Z((T-\lambda I)^h) \subset R(P)$ always, we see by (ii) of Lemma 7.1 that $\ell \leq h$. This shows that for all large $n$, we must have $\ell \leq \ell_{n,1} + \ldots + \ell_{n,k_n}$. //

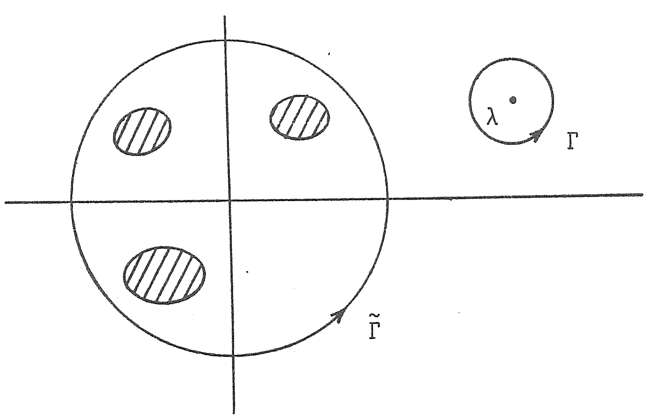
**REMARKS 14.6** (i) We give simple examples to show that in (14.14), $\ell$ need not equal $\ell_{n,1}+\ldots+\ell_{n,k_n}$ and $g$ need not equal $g_{n,1} + \ldots + g_{n,k_n}$ for all large $n$.

Let $X = \mathbb{C}^2$ and $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T_n = \begin{bmatrix} 1 & 0 \\ 1/n & 1 \end{bmatrix}$. Then $\|T_n - T\| \to 0$, $\lambda = 1$ is an eigenvalue of $T$ with $m = g = 2$ and $\ell = 1$, but the only eigenvalue $\lambda_n = 1$ of $T_n$ satisfies $m_n = 2$, $g_n = 1$, $\ell_n = 2$. Again, if we consider $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $T_n = \begin{bmatrix} 1 & 1 \\ 1/n & 1 \end{bmatrix}$, then $\|T_n - T\| \to 0$, $\lambda = 1$ is an eigenvalue of $T$ with $m = 2$, $g = 1$ and $\ell = 2$, while the eigenvalues $\lambda_{n,1} = 1 + \sqrt{1/n}$ and $\lambda_{n,2} = 1 - \sqrt{1/n}$ of $T_n$ satisfy $m_{n,j} = g_{n,j} = \ell_{n,j} = 1$ for $j = 1,2$.

Next, we give an example to show that if $\lambda_n \to \lambda$ and $\varphi_n$ is a corresponding eigenvector of $T_n$ of norm 1, then the sequence $(\varphi_n)$ itself (even after multiplying $\varphi_n$ by a constant of absolute value 1) may not converge to an eigenvector of $T$, although a subsequence of $(\varphi_n)$ must converge. let $X = \mathbb{C}^2$ and $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T_n = \begin{bmatrix} 1 & 1/n \\ 1/n & 1 \end{bmatrix}$. Then $\|T_n - T\| \to 0$, $\lambda = 1$ is an eigenvalue of $T$ with $m = g = 2$, while $\lambda_{n,1} = 1 + 1/n$ and $\lambda_{n,2} = 1 - 1/n$ are eigenvalues of $T_n$ . Let $\lambda_n = 1 + (-1)^n/n$ , and $\varphi_n = (1/\sqrt{2}\ ,\ (-1)^n/\sqrt{2})$ . Then $\varphi_n$ is an eigenvector of $T_n$ corresponding to $\lambda_n$ of norm 1 , the sequence $(c_n\varphi_n)$ , where $|c_n| = 1$ , has no limit, but the subsequence $(\varphi_{2n})$ converges to $(1/\sqrt{2}\ ,\ 1/\sqrt{2})$ , while the subsequence $(\varphi_{2n+1})$ converges to $(1/\sqrt{2}\ ,\ -1/\sqrt{2})$ , both of which are eigenvectors of $T$ corresponding to $\lambda = 1$ .

(ii) Let $\lambda$ be a discrete spectral value of $T$ , separated by a curve $\Gamma$ from the rest of $\sigma(T)$ as in Theorem 14.5. It is often of importance to know which eigenvalues of $T_n$ are close to $\lambda$ . We deal with a special case here. Let $\lambda$ be the *dominant spectral value of* $T$ . Then there is a circle $\tilde{\Gamma} \subset \rho(T)$ with centre $0$ such that

$$\sigma(T) \setminus \{\lambda\} \subset \operatorname{Int} \tilde{\Gamma} \quad \text{and} \quad \Gamma \subset \operatorname{Ext} \tilde{\Gamma} .$$




$////$ : $\sigma(T) \setminus \{\lambda\}$

Figure 14.1

For $n \geq n_0(\Gamma)$ , let $\lambda_{n,1},\ldots,\lambda_{n,k_n}$ be the eigenvalues of $T_n$ which lie inside $\Gamma$ , the sum of their algebraic multiplicities being equal to $m$ . Let, now, $n \geq n_0(\tilde{\Gamma})$ as well. Then $\tilde{\Gamma} \subset \rho(T_n)$ , and the spectral decomposition theorem (Theorem 6.3) shows that

$$\sigma(T_n) \cap \operatorname{Ext} \tilde{\Gamma} = \sigma(T_n|_{(I-\tilde{P}_n)(X)}) \ ,$$

where $\tilde{P}_n$ is the spectral projection associated with $T_n$ and $\tilde{\Gamma}$ . If $\tilde{P}$ is the spectral projection associated with $T$ and $\tilde{\Gamma}$ , then by Corollary 9.7,

$$\operatorname{rank}(I-\tilde{P}_n) = \operatorname{rank}(I-\tilde{P}) = \operatorname{rank} P = m ,$$

since $P + \tilde{P} = I$ . Hence $\lambda_{n,1},\ldots,\lambda_{n,k_n}$ are precisely the spectral points of $T_n$ outside $\tilde{\Gamma}$ . This shows that for $n \geq \max\{(n_0(\Gamma) , n_0(\tilde{\Gamma})\}$ ,

$$\begin{aligned}\sup\{|\mu| : \mu \in \sigma(T_n) , \mu \neq \lambda_{n,j} , j = 1,\ldots,k_n\} \\ < \min\{|\lambda_{n,j}| : j = 1,\ldots,k_n\} .\end{aligned}$$

Thus, if the dominant eigenvalue $\lambda$ of $T$ has algebraic multiplicity $m$ , then the $m$ eigenvalues of $T_n$ with largest moduli (counted according to their algebraic multiplicities) converge to $\lambda$ . This argument can be modified to treat the case where $T$ has no dominant spectral value, but

$$\sigma(T) \cap \{\lambda : |\lambda| = r_\sigma(T)\} \subset \sigma_d(T) .$$

A similar result for discrete spectral points of $T$ with second largest absolute value can also be proved. See [KY], and Problems 14.5, 14.6. These results are of great importance in choosing an initial approximation $\lambda_n$ of $\lambda$ in the iteration schemes we have developed in Section 11.

(iii) Let $\lambda$ be an isolated point of $\sigma(T)$ , and assume that $F = \{z \in \mathbb{C} : 0 < |z-\lambda| \leq \epsilon_0\} \subset \rho(T)$ for some $\epsilon_0 > 0$ . First, if $\lambda_n \in \sigma(T_n)$ and $|\lambda_n-\lambda| \leq \epsilon_0$ , then $\lambda_n \to \lambda$ . This can be seen as follows. Let $(\lambda_{n_k})$ be a subsequence of $(\lambda_n)$ such that $\lambda_{n_k} \to \mu$ . Then $|\mu-\lambda| \leq \epsilon_0$ . Also, $\mu \in \sigma(T)$ , for otherwise there is $r > 0$ such that $E = \{z \in \mathbb{C} : |z-\mu| \leq r\} \subset \rho(T)$ and hence for all

$n \geq n_0(E)$ , we have $E \subset \rho(T_n)$ , contradicting $\lambda_{n_k} \to \mu_n$ with $\lambda_{n_k} \in \sigma(T_{n_k})$ . Hence $\mu = \lambda$ . Thus, every convergent subsequence of the bounded sequence $(\lambda_n)$ converges to $\lambda$ , i.e., $\lambda_n \to \lambda$ . This proves the upper semicontinuity of $\sigma(T)$ at $\lambda$ . (Cf. Theorem 14.3.) Secondly, let $0 < \epsilon \leq \epsilon_0$ , and $\Gamma_\epsilon = \lambda + \epsilon e^{it}$ , $0 \leq t \leq 2\pi$ . Then for every $n \geq n_0(\Gamma_\epsilon)$ , there exists $\lambda_{n,\epsilon} \in \sigma(T_n)$ such that $|\lambda_{n,\epsilon}-\lambda| \leq \epsilon$ . This follows by noting that for $n \geq n_0(\Gamma_\epsilon)$ ,

$$\operatorname{rank} P_{\Gamma_\epsilon}(T_n) = \operatorname{rank} P_{\Gamma_\epsilon}(T) \neq 0 .$$

We thus have the lower semicontinuity of $\sigma(T)$ at $\lambda$ . (Cf. Theorem 14.5(b).)

(iv) We note that Theorem 14.5 (along with its proof) remains valid if we only assume that $T_n \xrightarrow{p} T$ and $\|(T-T_n)R(z)(T-T_n)\| \to 0$ for every $z \in \Gamma$ , in place of our overall assumption $T_n \xrightarrow{ro} T$ .

We now consider a very important special case.

Let $\lambda$ be a *simple* eigenvalue of $T$ , separated by a closed curve $\Gamma$ from the rest of $\sigma(T)$ . Let $\psi$ (resp., $\psi^*$) be an eigenvector of T (resp., $T^*$) corresponding to $\lambda$ (resp., $\bar{\lambda}$) such that $\|\psi\| = 1 = \langle\psi,\psi^*\rangle$ . Then the spectral projection $P$ associated with $T$ and $\lambda$ is given by

$$Px = \langle x,\psi^*\rangle\psi \ , \ x \in X .$$

Let $T_n \xrightarrow{ro} T$ . By Theorem 14.5, $\Gamma$ contains only one spectral value $\lambda_n$ of $T_n$ and it is a *simple* eigenvalue of $T_n$ for each $n \geq n_0(\Gamma)$ ; let $\varphi_n$ (resp., $\varphi_n^*$) be an eigenvector of $T_n$ (resp., $T_n^*$) corresponding to $\lambda_n$ (resp., $\bar{\lambda}_n$) such that $\|\varphi_n\| = 1 = \langle\varphi_n,\varphi_n^*\rangle$ . Then the spectral projection $P_n$ associated with $T_n$ and $\lambda_n$ is given by

$$P_n x = \langle x, \varphi_n^* \rangle \varphi_n \ , \quad x \in X \ .$$

Since $P_n \xrightarrow{p} P$ by Lemma 14.4, we have

$$(14.15) \qquad \langle \psi, \varphi_n^* \rangle \varphi_n = P_n \psi \longrightarrow P\psi = \psi \ .$$

In fact, if $\psi_n$ is any eigenvector of $T_n$ corresponding to $\lambda_n$, then there is a constant $c_n$ such that the entire sequence $(c_n \psi_n)$ (and not just a subsequence) converges to an eigenvector of $T$ corresponding to $\lambda$. In fact, since $\lambda_n$ is a simple eigenvalue of $T_n$ for all $n \geq n_0(\Gamma)$, we have $\langle \psi_n, \varphi_n^* \rangle \neq 0$, and if we let

$$c_n = \langle \psi, \varphi_n^* \rangle / \langle \psi_n, \varphi_n^* \rangle \ ,$$

then $c_n \psi_n = c_n P_n \psi_n = \langle \psi, \varphi_n^* \rangle \varphi_n \to \psi$.

Next, $\|\varphi_n\| = \|\psi\| = 1$, so that (14.15) implies

$$|\langle \psi, \varphi_n^* \rangle| \to 1 \ .$$

Taking scalar products with $\psi^*$ in (14.15), we also have

$$(14.16) \qquad \langle \psi, \varphi_n^* \rangle \langle \varphi_n, \psi^* \rangle \to \langle \psi, \psi^* \rangle = 1 \ .$$

Let $n_1$ be an integer such that

$$|\langle \psi, \varphi_n^* \rangle| \geq 1/2 \quad \text{for all} \quad n \geq n_1 \ ,$$

and let

$$(14.17) \qquad \varphi = \psi \ / \ \langle \psi, \varphi_n^* \rangle \ .$$

Then, $\varphi$ is an eigenvector of $T$ corresponding to $\lambda$; it depends on $n$ and satisfies

$$\langle \varphi, \varphi_n^* \rangle = 1 \ , \quad \|\varphi\| \leq 2 \ .$$

Note that by (14.6)

$$\sup\{\|R_n(z)\| : z \in \Gamma , n \geq n_0(\Gamma)\} = M(\Gamma) < \infty .$$

**THEOREM 14.7** Let $\lambda$ be a simple eigenvalue of $T$. With the notations introduced above, we have for all large $n$,

$$|\lambda-\lambda_n| \leq 2\|P_n\| \, \|(T-T_n)\psi\| \leq \frac{\ell(\Gamma)M(\Gamma)}{\pi} \|(T-T_n)P\| , \tag{14.18}$$

$$\|\varphi-\varphi_n\| = \|(P-P_n)\varphi\| \leq \frac{\ell(\Gamma)M(\Gamma)}{\pi \operatorname{dist}(\lambda,\Gamma)} \|(T-T_n)P\| . \tag{14.19}$$

**Proof** Noting that $|\langle\psi,\varphi_n^*\rangle| \geq 1/2$ for $n \geq n_1$, consider the operator $Q_n : P_n(X) \to P(X)$ given by

$$Q_n(t\varphi_n) = t\psi / \langle\psi,\varphi_n^*\rangle , \quad t \in \mathbb{C} . \tag{14.20}$$

Then it is clear that $\|Q_n\| \leq 2$, and

$$\begin{aligned} |\lambda-\lambda_n| &= \|(\lambda-\lambda_n)\psi\| \\ &= \|Q_nP_n(T-T_n)P\psi\| \\ &\leq 2\|P_n\| \, \|(T-T_n)P\| \\ &\leq \frac{\ell(\Gamma)M(\Gamma)}{\pi} \|(T-T_n)P\| , \end{aligned}$$

since $P_n = -\frac{1}{2\pi i}\int_\Gamma R_n(z)dz$. Thus, (14.18) is proved. Next, since $\langle\varphi,\varphi_n^*\rangle = 1$,

$$\begin{aligned} \varphi - \varphi_n &= (P-P_n)\varphi = (P-P_n)P\varphi , \\ P - P_n &= -\frac{1}{2\pi i}\int_\Gamma [R(z)-R_n(z)]dz \\ &= -\frac{1}{2\pi i}\int_\Gamma R_n(z)(T_n-T)R(z)dz . \end{aligned}$$

Since for the semisimple eigenvalue $\lambda$ we have $R(z)P = P/(\lambda-z)$ and $\|\varphi\| \leq 2$, we see that (14.19) holds. //

In Section 11, we have developed several iteration schemes for finding successive approximations to eigenelements $\lambda, \varphi$ of $T$ by starting with eigenelements $\lambda_0, \varphi_0$ of a 'nearby' operator $T_0$. We now wish to show that if $(T_n)$ is a resolvent operator appproximation of $T$ (which, of course, includes the important cases of the norm and the collectively compact approximations), then we can choose $T_0 = T_{n_0}$, where $n_0$ is fixed, and have the conditions for the convergence of the iteration schemes satisfied.

Before we consider other iterative schemes discussed in Section 11, we prove some preliminary results.

**PROPOSITION 14.8** Let $T_n \xrightarrow{ro} T$ on a closed subset $E$ of $\rho(T)$, and let $n \geq n_0(E)$. Then as $n \to \infty$,

(14.21) $$\max_{z \in E} \|(T-T_n)[R_n(z)-R(z)]\| \to 0 ,$$

(14.22) $$\max_{z \in E} \|(T-T_n)R_n(z)x\| \to 0 \quad \text{for every } x \in X ,$$

(14.23) $$\max_{z \in E} \|(T-T_n)R_n(z)(T-T_n)\| \to 0 ,$$

(14.24) $$\max_{z \in E} \|[(T-T_n)R_n(z)]^2\| \to 0 .$$

**Proof** For $n \geq n_0(E)$ and $z \in E$, let

$$A_n(z) = (T-T_n)[R_n(z)-R(z)] .$$

By (14.5), we have

$$(T-T_n)R_n(z) = (T-T_n)R(z)[I+(T-T_n)R(z)] \sum_{j=0}^{\infty} [(T-T_n)R(z)]^{2j} ,$$

so that

$$A_n(z) = [(T-T_n)R(z)]^2 + (T-T_n)R(z)[I+(T-T_n)R(z)] \sum_{j=1}^{\infty} [(T-T_n)R(z)]^{2j} .$$

Recalling $\upsilon_n(E) = \sup_{z\in E} \|(T-T_n)R(z)\|$ and $\delta_n(E) = \max_{z\in E} \|[(T-T_n)R(z)]^2\|$ , we see that

$$\|A_n(z)\| \le \delta_n(E) \left[1 + \frac{\upsilon_n(E)[1 + \upsilon_n(E)]}{1 - \delta_n(E)}\right] .$$

But by (14.2), $\upsilon_n(E) \le \upsilon < \infty$ and by (14.4), $\delta_n(E) \to 0$ . Hence

$$\max_{z\in E} \|A_n(z)\| \to 0 .$$

We have thus proved (14.21). Let now $x \in X$ . By Proposition 14.1, we have

$$\max_{z\in E} \|(T-T_n)R(z)x\| \to 0 .$$

Since $(T-T_n)R_n(z)x = A_n(z)x + (T-T_n)R(z)x$ , we see that (14.24) follows easily. Next, we note

$$(T-T_n)R_n(z)(T-T_n) = (T-T_n)[R_n(z)-R(z)](T-T_n) + (T-T_n)R(z)(T-T_n) ,$$
$$\|(T-T_n)R_n(z)(T-T_n)\| \le \|A_n(z)\|\|(T-T_n)\| + \|(T-T_n)R(z)(T-T_n)\| .$$

Now, $\max_{z\in E} \|A_n(z)\| \to 0$ , and $\max_{z\in E} \|(T-T_n)R(z)(T-T_n)\| \to 0$ by (14.3). Hence (14.23) holds. Since $\sup\{\|R_n(z)\| : n \ge n_0(\Gamma) , z \in E\} < \infty$ by (14.6), we see that (14.24) holds as well. //

**THEOREM 14.9** Let $\Gamma \subset \rho(T)$ separate a simple eigenvalue $\lambda$ of $T$ from the rest of $\sigma(T)$ . For $n \ge n_0(\Gamma)$ , let $\lambda_n$ be the simple eigenvalue of $T_n$ inside $\Gamma$ , and let $\varphi_n$ (resp., $\varphi_n^*$) be an eigenvector of $T_n$ (resp., $T_n^*$) corresponding to $\lambda_n$ (resp., $\bar{\lambda}_n$) such that $\|\varphi_n\| = 1 = \langle\varphi_n,\varphi_n^*\rangle$ . Let $S$ (resp., $S_n$) denote the reduced resolvent associated with $T$ and $\lambda$ (resp., $T_n$ and $\lambda_n$). Then

(14.25) $\qquad \|(T-T_n)\varphi_n\| \to 0$ ,

(14.26) $\qquad (\|\varphi_n^*\|)$ is a bounded sequence, (14.27) $S_n x \to Sx$ for every $x \in X$ ,

(14.28) $(\|S_n\|)$ and $(\|(T-T_n)S_n\|)$ are bounded sequences,

(14.29) $\|(T-T_n)S_n(T-T_n)\| \to 0$ , $\|[(T-T_n)S_n]^2\| \to 0$ .

**Proof** Since

$$\|(T-T_n)\varphi_n\| = \|(T-T_n)P_n\varphi_n\| \leq \|(T-T_n)P_n\|$$

and $\|(T-T_n)P_n\| \to 0$ by (14.9), we see that $\|(T-T_n)\varphi_n\| \to 0$ . Next, since

$$P_n x = \langle x, \varphi_n^* \rangle \varphi_n$$

and $\|\varphi_n\| = 1$ , we see that $\|\varphi_n^*\| = \|P_n\|$ . But $P_n \xrightarrow{p} P$ . Hence by the uniform boundedness principle, $(\|\varphi_n^*\|)$ is a bounded sequence.

Let $x \in X$ . By (14.7) $R_n(z)x \to R(z)x$ uniformly for $z \in \Gamma$ . Also, since $\lambda_n \to \lambda$ by (14.12), we see that $\operatorname{dist}(\lambda_n, \Gamma) \geq \operatorname{dist}(\lambda, \Gamma)/2 > 0$ for all large $n$ . Hence

$$S_n x = \frac{1}{2\pi i} \int_\Gamma \frac{R_n(z)x}{z - \lambda_n} dz \longrightarrow \frac{1}{2\pi i} \int_\Gamma \frac{R(z)x}{z - \lambda} dz = Sx .$$

The sequences $(\|S_n\|)$ and $(\|(T-T_n)S_n\|)$ are then bounded by the uniform boundedness principle. Lastly, to prove (14.29) we note that

$$(T-T_n)S_n(T-T_n) = \frac{1}{2\pi i} \int_\Gamma \frac{(T-T_n)R_n(z)(T-T_n)}{z - \lambda_n} dz .$$

Again, since $\operatorname{dist}(\lambda_n, \Gamma) > \operatorname{dist}(\lambda, \Gamma)/2 > 0$ , we see by (14.23) that $\|(T-T_n)S_n(T-T_n)\| \to 0$ . Since $(\|S_n\|)$ is a bounded sequence, we also have $\|[(T-T_n)S_n]^2\| \to 0$ . //

**REMARK 14.10** We are now in a position to show that various conditions needed for the convergence of the iteration schemes and for obtaining error estimates considered in Sections 10 and 11 can be fulfilled if we choose $T_0 = T_{n_0}$ for some suitable $n_0$, when $T_n \xrightarrow{ro} T$. Let a simple eigenvalue $\lambda$ of $T$ be isolated from the rest of $\sigma(T)$ by a simple closed rectifiable curve $\Gamma$ in $\rho(T)$, and let $n \geq n_0(\Gamma)$. Then by Theorem 14.5(b), the only spectral value of $T_n$ inside $\Gamma$ is a simple eigenvalue $\lambda_n$. Since

$$\max_{z\in\Gamma} r_\sigma((T-T_n)R_n(z)) \leq \max_{z\in\Gamma} \|[(T-T_n)R_n(z)]^2\|^{1/2} \to 0$$

by (14.24), we can choose $n_0 \geq n_0(\Gamma)$ such that $\max_{z\in\Gamma} \|[(T-T_{n_0})R(z)]^2\| < 1$. Hence

$$\max_{z\in\Gamma} r_\sigma((T-T_{n_0})R_{n_0}(z)) < 1 .$$

Then the Rayleigh-Schrödinger series (10.4) for the eigenvalue $\lambda(t)$ of $T(t) = T_{n_0} + t(T-T_{n_0})$ (with initial term $\lambda_{n_0}$) converges for $|t| \leq 1$, because $1 \in \partial_\Gamma$. (See (9.14).) In particular, putting $t = 1$, we see that $\lambda(1)$ is the only spectral value of $T(1) = T$ inside $\Gamma$ and it is a simple eigenvalue. Hence it must coincide with the eigenvalue $\lambda$ we started with. Thus,

$$\lambda = \lambda_{n_0} + \sum_{k=1}^{\infty} \lambda_{(k)} .$$

As for the convergence of the Rayleigh-Schrödinger series (10.7) for the eigenvector $\varphi(1)$ of $T(1) = T$, it is sufficient to have

$$a_n + (a_n+b_n)c_n < 1 ,$$

where

$$a_n = \max_{z\in\Gamma} \|[(T-T_n)R_n(z)]^2\| ,$$

$$b_n = \max_{z\in\Gamma} \|(T-T_n)R_n(z)\| ,$$

$$c_n = \frac{\ell(\Gamma)\|(T-T_n)\varphi_n\| \, \|\varphi_n^*\|}{2\pi[\operatorname{dist}(\lambda_n,\Gamma)]^2} .$$

(See Proposition 10.2.) Now, the relations (14.24), (14.6), (14.25), (14.26) and the fact that $\lambda_n \to \lambda$ as $n \to \infty$, show that $a_n \to 0$, $(b_n)$ is a bounded sequence and $c_n \to 0$. Hence we can choose $n_0$ so that $a_{n_0} + (a_{n_0}+b_{n_0})c_{n_0} < 1$, and ensure the convergence of the eigenvector series

$$\varphi(1) = \varphi_{n_0} + \sum_{k=1}^{\infty} \varphi_{(k)} .$$

Next, let

$$\begin{aligned} &\eta_n = \|(T-T_n)\varphi_n\| , \quad p_n = \|\varphi_n^*\| , \quad s_n = \|S_n\| , \\ &\alpha_n = \|(T-T_n)S_n\| , \quad \beta_n = \eta_n p_n s_n , \quad \gamma_n = \max\{\alpha_n , \beta_n\} , \\ &\epsilon_n = \max\{\|[(T-T_n)S_n]^2\| , \beta_n^2 , \alpha_n^{3/2}\beta_n^{1/2}\} . \end{aligned}$$

Then by (14.25), (14.26), (14.28) and (14.29) we see that $\eta_n \to 0$, $(p_n)$, $(s_n)$ and $(\alpha_n)$ are bounded sequences, $\beta_n \to 0$ and $\epsilon_n \to 0$. For the estimates given in (11.30) for the Rayleigh-Schrödinger iteration scheme (11.18) as well as the fixed point iteration scheme (11.19), we need $\sqrt{\epsilon_n} < 1/4$. This can now very well be achieved for a convenient value $n_0$ of $n$. Then, as pointed out in Remark 11.9(v), we have better bounds for the successive iterates at *every other* step. In case $T_n \xrightarrow{\| \|} T$, we have $\alpha_n \to 0$, and hence $\gamma_n \to 0$. If, in this case, we choose $n_0$ such that $\gamma_{n_0} < 1/4$, then Theorem 11.5 shows that better bounds for the iterates are available at *every* step. Thus, we have a geometric convergence of the iterates, as against the semigeometric convergence when we only have $\sqrt{\epsilon_{n_0}} < 1/4$. (See Table 19.4, Rayleigh-Schrödinger and fixed point schemes.)

If $\gamma_{n_0} < 1/4$ (resp., $\sqrt{\epsilon_{n_0}} < 1/4$), then Theorem 11.8 (resp., Problem 11.2) shows that both the iteration schemes (11.18) and (11.19) converge to a simple eigenvalue $\mu$ of $T$, and $\mu$ is the closest spectral point of $T$ to $\lambda_{n_0}$ . Thus, if the simple eigenvalue $\lambda$ of $T$ with which we started is the closest spectral value of $T$ from $\lambda_0$ , then $\mu = \lambda$ . Note that this can be achieved by taking $n_0$ large enough since $\lambda_n \to \lambda$ as $n \to \infty$ . Another way of achieving this is to choose $\Gamma$ to be a circle with radius $\leq \text{dist}(\lambda,\sigma(T)\setminus\{\lambda\})/2$ and centre $\lambda$ , and find an eigenvalue $\lambda_{n_0}$ of $T_{n_0}$ inside $\Gamma$ .

Next, assume that $\lambda$ is the dominant (simple) eigenvalue of $T$ . Since $r_\sigma(P_n(P-P_n)) \to 0$ by (14.10), we can choose $n_0$ such that

$$r_\sigma(P_{n_0}(P-P_{n_0})) < 1 . \tag{14.30}$$

Remark 11.13(a) shows that we can let $x_0 = \varphi_{n_0}$ and $x_0^* = \varphi_{n_0}^*$ as the starting vectors in the power method (Note: $\ell = 1$ , $D^{\ell-1} = P$ and $\langle P\varphi_{n_0}, \varphi_{n_0}^* \rangle \neq 0$ ) .

Finally, let $\lambda \neq 0$ . Then by Theorem 14.7, for <u>all large n</u>

$$|\lambda-\lambda_n| \ , \ \|\varphi-\varphi_n\| \leq c\|(T-T_n)P\| \ ,$$

where the eigenvector $\varphi$ of $T$ depends on $n$ , and

$$c = \frac{\ell(\Gamma)M(\Gamma)}{\pi} \max\{1 \ , \ \frac{1}{\text{dist}(\lambda,\Gamma)}\} \ ,$$
$$M(\Gamma) = \sup\{\|R_n(z)\| \ : \ z \in \Gamma \ , \ n \geq n_0(\Gamma)\} < \infty \ .$$

Also, by (14.9), $\|(T-T_n)P\| \to 0$ . Now, assume that $T$ is compact. Then $\|(T-T_n)T\| \to 0$ . Thus, we see that the modified fixed point iteration scheme (11.31) would converge to $\varphi$ and $\lambda$ if we choose $T_0 = T_{n_0}$ for a suitably large $n_0$ . This remark holds for the Ahués iteration scheme (11.35) as well. The error bounds (11.34) are in terms of $\|(T-T_{n_0})T\|$ and they improve at *every* step.

**Problems**

14.1 Let $A_n$, $B_n$, $T \in BL(X)$ be such that $\|A_n\|$ and $\|B_n\|$ are bounded. Then

$$\|A_n R(z) B_n\| \to 0 \ , \quad \text{as} \quad n \to \infty$$

for all $z$ in the unbounded conncected component of $\rho(T)$ if and only if

$$\|A_n T^k B_n\| \to 0 \ , \quad \text{as} \quad n \to \infty$$

for each fixed $k = 0,1,2,\ldots$ . (Hint: Problem 4.3, (5.8) and (5.9))

14.2 Let $E \subset \rho(T)$ be a compact set, and let $\Gamma$ be a simple closed rectifiable curve in $\rho(T)$ . (a) Let $T_n \xrightarrow{\|\ \|} T$ . Then $R_n(z) \xrightarrow{\|\ \|} R(z)$ uniformly for $z \in E$ and $P_\Gamma(T_n) \xrightarrow{\|\ \|} P_\Gamma(T)$ . (b) Let $T_n \xrightarrow{cc} T$ . Then $R_n(z) \xrightarrow{cc} R(z)$ uniformly for $z \in E$ (i.e., $R_n(z) \xrightarrow{p} R(z)$ uniformly for $z \in E$ , and for some positive integer $n_0$ , the set

$$\bigcup_{n=n_0}^{\infty} \ \bigcup_{z \in E} \{R_n(z)x - R(z)x : x \in X \ , \ \|x\| \le 1\}$$

has a compact closure in $X$ ) and $P_\Gamma(T_n) \xrightarrow{cc} P_\Gamma(T)$ . (Hint: Proposition 4.2 of [AN]. Compare this result with Lemma 14.4, where the rank of $P_\Gamma(T)$ is assumed to be finite.)

14.3 Let $\Gamma \subset \rho(T)$ , rank $P_\Gamma(T) < \infty$ and $T_n \xrightarrow{ro} T$ . Then for all large $n$ , we have $\Gamma \subset \rho(T_n)$ , and the arithmetic mean of the eigenvalues of $T_n$ inside $\Gamma$ converges to the arithmetic mean of the eigenvalues of $T$ inside $\Gamma$ . Let $x_n \in P_\Gamma(T_n)$ and $\|x_n\| = 1$ . Then $\|x_n - Px_n\| \to 0$ , $\|Px_n\| \to 1$ , and there is a subsequence $(x_{n_k})$ such that $x_{n_k} \to x$ for some $x \in R(P_\Gamma)$ with $\|x\| = 1$ .

14.4 With the notations of Theorem 14.5,

$$\sum_{j=1}^{k_n} \ell_{n,j} \leq m + k_n - \sum_{j=1}^{k_n} g_{n,j} \leq m .$$

14.5 Let $T \in BL(X)$ and assume that

$$\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = r_\sigma(T)\} = \{\lambda,\mu\} \subset \sigma_d(T) ,$$

with $\lambda \neq \mu$ . The considerations of Remark 14.6(ii) can be extended to $\lambda$ and $\mu$ .

14.6 Let $T \in BL(X)$ , $\lambda$ be the dominant discrete spectral value of $T$ and let $\mu$ be a discrete spectral value of $T$ such that if $\tilde{\lambda} \in \sigma(T)$ , $\tilde{\lambda} \neq \lambda$ , $\tilde{\lambda} \neq \mu$ , then $|\tilde{\lambda}| < |\mu|$ . Then Remark 14.6(ii) can be extended to $\mu$ .

4.7 With the notations of Theorem 14.7, $|\langle\varphi_n,\psi^*\rangle| \to 1$ . Choose $\tilde{n}_1$ such that $n \geq \tilde{n}_1$ implies $|\langle\varphi_n,\psi^*\rangle| \geq 1/2$ . Let $\tilde{M}(\Gamma) = \sup\{\|R(z)\| : z \in \Gamma\}$ . Then

$$|\lambda-\lambda_n| \leq 2\|P\| \; \|(T-T_n)\varphi_n\| \leq \frac{\ell(\Gamma)\tilde{M}(\Gamma)}{\pi} \|T\varphi_n-\lambda_n\varphi_n\| ,$$

$$\|\varphi_n-P\varphi_n\| = \|(P_n-P)\varphi_n\| \leq \frac{\ell(\Gamma)\tilde{M}(\Gamma)}{2\pi \operatorname{dist}(\lambda_n,\Gamma)} \|(T-T_n)\varphi_n\| .$$

14.8 Let $\lambda \in \sigma_d(T)$ be isolated by a closed curve $\Gamma$ . If $T_n \xrightarrow{ro} T$ , then $(\|(T-T_n)S_\lambda\|)$ is bounded and $\|(T-T_n)S_\lambda(T-T_n)\| \to 0$ .