7. ISOLATED SINGULARITIES OF $R(z)$

In the last section we have considered the Laurent expansion of the resolvent operator $R(z)$ in an annulus contained in the resolvent set $\rho(T)$ of $T \in BL(X)$. We now specialize to the case when the inner circle of such an annulus degenerates to a point λ; i.e., when a punched disk $\{z \in \mathbb{C} : 0 < |z-\lambda| < \delta\}$ lies in $\rho(T)$. Let Γ be any curve in $\rho(T)$ such that $\sigma(T) \cap \operatorname{Int} \Gamma \subset \{\lambda\}$. Since the operators $P_\Gamma(T)$, $S_\Gamma(T,\lambda)$ and $D_\Gamma(T,\lambda)$ do not depend on Γ, we denote them simply by P_λ, S_λ and D_λ, respectively. The operators S_λ and D_λ have special features. By the first resolvent identity (5.5), we have

$$\begin{aligned}
S_\lambda &= \frac{1}{2\pi i}\int_\Gamma \frac{R(w)}{w-\lambda}\,dw \\
&= \lim_{z\to\lambda} \frac{1}{2\pi i}\int_\Gamma \frac{R(w)}{w-z}\,dw \\
&= \lim_{z\to\lambda} \frac{1}{2\pi i}\int_\Gamma \frac{R(z)+R(w)-R(z)}{w-z}\,dw \\
&= \lim_{z\to\lambda} \frac{1}{2\pi i}\left[R(z)\int_\Gamma \frac{dw}{w-z} + \int_\Gamma \frac{(w-z)R(z)R(w)}{w-z}\,dw\right] \\
&= \lim_{z\to\lambda}\left[R(z) + R(z)(-P)\right].
\end{aligned}$$

Thus, we see that

$$S_\lambda = \lim_{z\to\lambda} R(z)(I-P). \tag{7.1}$$

Next, it follows by Proposition 6.4 and (5.1) that

$$\sigma(S_\lambda) \subset \{0\} \cup \{1/(\mu-\lambda) : \mu \in \sigma(T),\ \mu \neq \lambda\}, \tag{7.2}$$

where the inclusion is proper if and only if $\lambda \notin \sigma(T)$. Hence

$$r_\sigma(S_\lambda) = \frac{1}{\operatorname{dist}(\lambda, \sigma(T)\setminus\{\lambda\})}. \tag{7.3}$$

Again, Proposition 6.4 implies that

$$\sigma(D_\lambda) = \{0\} \text{ and } r_\sigma(D_\lambda) = 0. \tag{7.4}$$

For this reason, the operator D_λ will be called the <u>quasi-nilpotent operator associated with</u> T <u>and</u> λ. We thus have the representation

(7.5) $$T|_{P_\lambda(X)} = \lambda I|_{P_\lambda(X)} + D_\lambda|_{P_\lambda(X)} ,$$

where D_λ is quasi-nilpotent.

For $0 < |z-\lambda| < \mathrm{dist}(\lambda,\sigma(T)\setminus\{\lambda\})$, we have the Laurent expansion

(7.6) $$R(z) = \sum_{k=0}^{\infty} S_\lambda^{k+1}(z-\lambda)^k - \frac{P_\lambda}{z-\lambda} - \sum_{k=1}^{\infty} \frac{D_\lambda^k}{(z-\lambda)^{k+1}}$$

To have a feeling for the operators P_λ and S_λ, we give a simple example. Let T be represented by the diagona' matrix

$$\mathrm{diag}(\lambda,\ldots,\lambda,\lambda_1,\lambda_2,\ldots) ,$$

where λ does not belong to the closure of $\{\lambda_j : j = 1,2,\ldots\}$. Then

$$P_\lambda = \mathrm{diag}(1,\ldots,1,0,0,\ldots) ,$$
$$S_\lambda = \mathrm{diag}(0,\ldots,0,1/(\lambda_1-\lambda) , 1/(\lambda_2-\lambda),\ldots) .$$

Let us consider another typical example. Let $X = L^2([a,b])$ and let V denote the <u>Volterra integration operator</u> defined by

$$Vx(s) = \int_a^s x(t)dt , \ x \in X , \ s \in [a,b] .$$

Then it is well-known ([L], p.151) that

$$\sigma(V) = \{0\} ,$$

i.e., V is quasi-nilpotent. Also, $Vx = 0$ implies $x = 0$, since $\int_0^s x(t)dt = 0$ for almost all $s \in [a,b]$ implies that $x(t) = 0$ for almost all $t \in [a,b]$. Thus, 0 is not an eigenvalue of V. Hence V is not nilpotent.

Considering the isolated spectral point $\lambda = 0$ of V, we easily see that

$$P_0 = I\ ,\quad D_0 = (V-0I)P_0 = V \quad \text{and}$$
$$S_0 = \lim_{z\to 0} R(z)(I-P_0) = 0\ .$$

This confirms with the first Neumann expansion (5.8)

$$R(z) = -\sum_{k=0}^{\infty} V^k z^{-(k+1)}$$
$$= -\frac{I}{z} - \sum_{k=1}^{\infty} \frac{V^k}{z^{k+1}}\ ,$$

for $0 \neq z \in \mathbb{C}$, which is also the Laurent expansion (6.22) about 0 of $R(z)$.

It can be readily seen by induction that for each $k \geq 1$,

$$V^k x(s) = \int_a^s \frac{(s-t)^{k-1}}{(k-1)!} x(t)dt\ ,\quad x \in X\ ,\ s \in [a,b]\ .$$

Hence, if we let for $0 \neq z \in \mathbb{C}$,

$$U(z)x(s) = \int_a^s \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left[\frac{s-t}{z}\right]^{k-1} x(t)dt\ ,$$
$$= \int_a^s e^{(s-t)/z} x(t)dt\ ,\quad x \in X\ ,\ s \in [a,b]\ ,$$

then

$$R(z) = -I/z - U(z)/z^2\ ,$$

where $U(z)$ is again a Volterra operator with kernel $e^{(s-t)/z}$.

The above remarks and the infinite representation of $R(z)$ hold for any quasi-nilpotent operator which is not nilpotent.

In the above example $\lambda = 0$ is an *isolated essential singularity* of $R(z)$, since the Laurent expansion (7.6) has infinitely terms with negative powers of $(z-\lambda)$.

The other extreme case arises when λ is a *removable singularity* of $R(z)$, so that there are no terms with negative powers of $(z-\lambda)$ in (7.6). Clearly, this happens if and only if $P_\lambda = 0$, i.e., $\lambda \notin \sigma(T)$ (Proposition 6.4(a)). In this case, $S_\lambda = R(\lambda)$ and we recover the Taylor expansion (5.7) of $R(z)$ around λ:

$$R(z) = \sum_{k=0}^{\infty} R(\lambda)^{k+1}(z-\lambda)^k .$$

Let us now consider the important case where λ is a *pole* of $R(z)$. It can be readily seen from (7.6) that λ is a pole of order ℓ, $1 \leq \ell < \infty$, if and only if

$$D_\lambda^{\ell-1} \neq 0 , \quad \text{but} \quad D_\lambda^{\ell} = 0 . \tag{7.7}$$

In this case (7.6) reduces to

$$R(z) = \sum_{k=0}^{\infty} S_\lambda^{k+1}(z-\lambda)^k - \frac{P_\lambda}{z-\lambda} - \sum_{k=1}^{\ell-1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}} , \tag{7.8}$$

where $D_\lambda^{\ell-1} \neq 0$, with the notation $D_\lambda^0 = P_\lambda$. Notice that $-P_\lambda$ is the *residue of* $R(z)$ *at* λ and that D_λ is *nilpotent*..

In order to illustrate the calculation of the coefficients in the expansion (7.8) of $R(z)$, we consider a simple example. Let $X = \mathbb{C}^2$ and fix $t \in \mathbb{C}$. Let

$$T(t)x = \begin{bmatrix} 0 & t/16 \\ 4t & 2 \end{bmatrix} \begin{bmatrix} x(1) \\ x(2) \end{bmatrix}$$

for $x = [x(1),x(2)]^t \in \mathbb{C}^2$. Then for $z \in \mathbb{C}$,

$$\det(T(t)-zI) = -z(2-z) - t^2/4 .$$

Let

$$\lambda(t) = \frac{2+\sqrt{4+t^2}}{2} , \quad \mu(t) = \frac{2-\sqrt{4+t^2}}{2} ,$$

where $\sqrt{4+t^2}$ denotes the principle value of the square root of $4+t^2$. Then every $z \notin \{\lambda(t), \mu(t)\}$ lies in $\rho(T(t))$, and $R(T(t),z)$ is given by the matrix

$$(T(t)-zI)^{-1} = \begin{bmatrix} 2-z & -t/16 \\ -4t & -z \end{bmatrix} / [z-\lambda(t)][z-\mu(t)] .$$

Note that $R(T(t),z)$ has simple poles at $z = \lambda(t)$ and $z = \mu(t)$ if $t \neq \pm 2i$, and if $t = \pm 2i$, then it has a double pole at $z = 1$. Let Γ denote the circle $\Gamma(t) = 2 + e^{it}$, $0 \leq t \leq 2\pi$. Since for $|t| < 2$, we have $|1 - \sqrt{1+t^2/4}| < 1$ and $|1 + \sqrt{1+t^2/4}| > 1$, we see that $\lambda(t)$ lies inside Γ and $\mu(t)$ lies outside Γ. Using Cauchy's integral formula (Theorem 4.5(b)), we see that for $|t| < 2$,

$$P_{\lambda(t)} = P_\Gamma(T(t)) = -\frac{1}{2\pi i}\int_\Gamma R(T(t),z)dz$$

is given by the matrix

$$\begin{bmatrix} \lambda(t)-2 & t/16 \\ 4t & \lambda(t) \end{bmatrix} / (\lambda(t)-\mu(t)) .$$

It can be readily checked that for $|t| < 2$,

$$D_{\lambda(t)} = (T(t)-\lambda(t)I)P_{\lambda(t)} = 0 .$$

Also, $S_{\lambda(t)} = \lim_{z\to\lambda(t)} R(T(t),z)(I-P_{\lambda(t)})$ is given by the matrix

$$\begin{bmatrix} -\lambda(t) & t/16 \\ 4t & \mu(t) \end{bmatrix} / (4+t^2) .$$

Now we prove a result which allows us to characterize the order of a pole of $R(z)$.

LEMMA 7.1 Let λ be an isolated point of $\sigma(T)$.

(a) For $k = 1,2,\ldots,$ $D_\lambda^k = 0$ if and only if $Z(P_\lambda) = R((T-\lambda I)^k)$ if and only if $R(P_\lambda) = Z((T-\lambda I)^k)$.

(b) Let $1 \leq \ell < \infty$. Then λ is a pole of $R(z)$ of order ℓ if and only if ℓ is the smallest positive integer such that one (and hence each) of the following conditions holds:

(i) $Z(P_\lambda) = R((T-\lambda I)^\ell)$
(ii) $R(P_\lambda) = Z((T-\lambda I)^\ell)$

In that case,

$$X = Z((T-\lambda I)^\ell) \oplus R((T-\lambda I)^\ell) .$$

Proof (a) Let $k = 1,2,\ldots$. We have already noted in Section 6 (just before the definition of a spectral projection) that

$$Z((T-\lambda I)^k) \subset R(P_\lambda) \tag{7.9}$$

Similarly, it follows (cf. Problem 6.2) that

$$R((T-\lambda I)^k) \supset Z(P_\lambda) . \tag{7.10}$$

Also, since $(T-\lambda I)$ and P_λ commute, we have

$$D_\lambda^k = (T-\lambda I)^k P_\lambda = P_\lambda (T-\lambda I)^k .$$

Hence part (a) follows.

(b) It is clear from part (a) that $D_\lambda^{\ell-1} \neq 0$ and $D^\ell = 0$ if and only if (i) or (ii) holds and ℓ is the smallest such positive integer. In that case,

$$X = R(P_\lambda) \oplus Z(P_\lambda) = Z((T-\lambda I)^\ell) \oplus R((T-\lambda I)^\ell) . \qquad //$$

Remark 7.2 Consider the following two chains of inclusions involving the null spaces and the range spaces of powers of an operator A :

$$\{0\} \subset Z(A) \subset Z(A^2) \subset \dots$$

$$X \supset R(A) \supset R(A^2) \supset \dots$$

A peculiar property of each of these chains is that if equality holds at any inclusion then it persists at all later inclusions. This can be seen as follows. Let $Z(A^k) = Z(A^{k+1})$. If $x \in Z(A^{k+2})$, then $A^{k+1}(Ax) = 0$, i.e., $Ax \in Z(A^{k+1}) = Z(A^k)$, or $A^{k+1}x = 0$. Thus, $Z(A^{k+1}) = Z(A^{k+2})$. Similarly, let $R(A^k) = R(A^{k+1})$. If $y \in R(A^{k+1})$, then $y = A(A^k x)$ for some $x \in X$; but $A^k x \in R(A^k) = R(A^{k+1})$, i.e., $A^k x = A^{k+1}x_0$ or $y = A^{k+2}x_0$ for some $x_0 \in X$. Thus $R(A^{k+1}) = R(A^{k+2})$. We shall make use of this property frequently. See Theorem 2 of Appendix I for a characterization of a pole of $R(z)$.

Here is an iterative procedure for finding $Z(A^k)$: Let

$$Z_0 = \{0\}\ ,\ Z_1 = Z(A) \setminus Z_0\ , \quad \text{and } Z_k = \{x \in X : Ax \in Z_{k-1}\}\ ,\quad k = 2,3,\dots \tag{7.11}$$

Then it is easy to see by induction on k that

$$Z_k = Z(A^k) \setminus Z(A^{k-1})$$

for all k , i.e., Z_k consists of the generalized eigenvectors of A of grade k corresponding to 0 . In particular, $Z_k = \emptyset$ if and only if $Z(A^{k-1}) = Z(A^k)$. We have the disjoint union $Z(A^k) = Z_0 \cup \dots \cup Z_k$.

PROPOSITION 7.3 Let λ be a pole of $R(z)$. Then λ is an isolated eigenvalue of T , and the associated spectral subspace $R(P_\lambda)$ coincides with the generalized eigenspace of T corresponding to λ . In fact, the order of the pole of $R(z)$ at λ is ℓ if and only if ℓ

is the smallest positive integer such that there are no generalized eigenvectors of T of grade $\ell + 1$ corresponding to λ, and in that case $R(P_\lambda)$ is the disjoint union of $\{0\}$ and the sets of generalized eigenvectors of T of grade k corresponding to λ, $k = 1,\ldots,\ell$.

Proof Since λ is a pole of $R(z)$, we have $D_\lambda^\ell = 0$, but $D_\lambda^{\ell-1} \neq 0$ for some positive integer ℓ. Then there is $D_\lambda^{\ell-1}x \neq 0$ with

$$(T-\lambda I)D_\lambda^{\ell-1}x = D_\lambda^\ell x = 0 .$$

Thus, $D_\lambda^{\ell-1}x$ is an eigenvector of T corresponding to the eigenvalue λ. By (ii) of Lemma 7.1(b) and by (7.9), we have

$$\begin{aligned} R(P_\lambda) &= Z((T-\lambda I)^\ell) \\ &= \{x \in X : (T-\lambda I)^k x = 0 \text{ for some } k = 1,2,\ldots\} \end{aligned}$$

Letting $A = T - \lambda I$ in (7.11), we have

$$Z_k = Z((T-\lambda I)^k) \setminus Z((T-\lambda I)^{k-1}) ,$$

and hence

$$R(P_\lambda) = Z_0 \cup \ldots \cup Z_\ell .$$

where $Z_i \cap Z_j = \emptyset$ if $i \neq j$. Also, for $k \geq 1$, $Z_{k+1} = \emptyset$ if and only if $Z((T-\lambda I)^k) = Z((T-\lambda I)^{k+1})$, and this is the case if and only if $R(P_\lambda) = Z((T-\lambda I)^k)$. Thus, λ is a pole of order ℓ if and only if ℓ is the smallest positive integer with $Z_{\ell+1} = \emptyset$. //

When λ is a pole of $R(z)$, we wish to investigate how much larger the generalized eigenspace $P_\lambda(X)$ is as compared to the eigenspace $Z(T-\lambda I)$. For this purpose we prove the following result. It will also allow us to obtain necessary and sufficient conditions for the spectral projection P_λ to be of finite rank.

LEMMA 7.4 Let A be a linear operator on X. Then for $k = 1,2,\ldots,$

$$\dim Z(A^k) \leq \dim Z(A) + \dim Z(A^{k-1}) \leq k \dim Z(A) . \tag{7.12}$$

If $Z(A^k) \setminus Z(A^{k-1}) = Z_k \neq \emptyset$, then

$$\dim Z(A) + k - 1 \leq \dim Z(A^k) . \tag{7.13}$$

Proof Since $Z(A^{k-1}) \subset Z(A^k)$, let us extend a basis of $Z(A^{k-1})$ to a basis of $Z(A^k)$ by adding a set W to it. Let $x_1,\ldots,x_n \in W$. Then $A^{k-1}x_1,\ldots,A^{k-1}x_n \in Z(A)$, and they form a linearly independent set. This can be seen as follows. Let

$$0 = c_1A^{k-1}x_1 + \ldots + c_nA^{k-1}x_n = A^{k-1}(c_1x_1 + \ldots + c_nx_n)$$

for some $c_1,\ldots,c_n$ in $\mathbb{C}$. Then $x = c_1x_1 + \ldots + c_nx_n \in Z(A^{k-1})$, and since $x_1,\ldots,x_n$ belong to W, we must have $c_1 = \ldots = c_n = 0$. Thus, $n \leq \dim Z(A)$. This shows that

$$\dim Z(A^k) \leq \dim Z(A) + \dim Z(A^{k-1}) .$$

Applying this result repeatedly for $k = 2,3,\ldots,$ we obtain (7.12).

Next, assume that $Z(A^{k-1}) \neq Z(A^k)$. Then by Remark 7.2, each inclusion in the chain

$$Z(A) \subset Z(A^2) \ldots \subset Z(A^{k-1}) \subset Z(A^k)$$

is proper. Hence

$$\dim Z(A) + 1 + \ldots + 1 \leq \dim Z(A^k) ,$$

where the 1's occur (k-1) times. This proves (7.13). //

THEOREM 7.5 (a) Let λ be a pole of $R(z)$ of order ℓ. If m is the rank of P_λ, and g is the dimension of the eigenspace of T corresponding to λ, then

$$m \leq \ell g ,$$
$$2 \leq \ell + g \leq m + 1 . \tag{7.14}$$

In particular,

$$\begin{aligned} m = 1 &\text{ if and only if } \ell = 1 = g \\ g = 1 &\text{ if and only if } m = \ell \\ \ell = 1 &\text{ if and only if } m = g . \end{aligned} \tag{7.15}$$

(b) For an isolated point λ of $\sigma(T)$, we have $\operatorname{rank} P_\lambda < \infty$ if and only if λ is a pole of $R(z)$ and $\dim Z(T-\lambda I) < \infty$.

Proof (a) By Lemma 7.1(b), we have $R(P_\lambda) = Z((T-\lambda I)^\ell)$. Hence letting $A = T - \lambda I$ in (7.12) we see that

$$m = \dim R(P_\lambda) \leq \ell \dim Z(T-\lambda I) = \ell g .$$

Proposition 7.3 shows that λ is an eigenvalue of T. Hence $g \geq 1$. Since $\ell \geq 1$, we have $2 \leq \ell + g$. Again, since $D^{\ell-1} \neq 0$, but $D^\ell = 0$, we have $Z((T-\lambda I)^{\ell-1}) \neq Z((T-\lambda I)^\ell)$. Hence by (7.13),

$$g + \ell - 1 = \dim Z(T-\lambda I) + \ell - 1 \leq \dim Z((T-\lambda I)^\ell) = m .$$

This proves (7.14). The relations in (7.15) are immediate.

(b) Assume that $\operatorname{rank} P_\lambda = m < \infty$. As we have seen in (7.4), D_λ is quasi-nilpotent. Since $Y = P_\lambda(X)$ is of dimension m, we see by Proposition 5.6 that $(D_\lambda|_Y)^m = 0$. Also $D_\lambda|_Z = 0$, where $Z = Z(P_\lambda)$. Hence $D_\lambda^m = 0$, showing that λ is a pole of $R(z)$.

Since $Z(T-\lambda I) \subset P_\lambda(X)$, it follows that

$$g = \dim Z(T-\lambda I) \leq m < \infty .$$

Conversely, let λ be a pole of $R(z)$ of order ℓ and let $g = \dim Z(T-\lambda I) < \infty$. Then by (7.14) we see that rank $P_\lambda < \infty$. //

Let λ be an isolated point of $\sigma(T)$. The dimension of the associated spectral subspace $P_\lambda(X)$ is called the <u>algebraic multiplicity</u> of λ , and the dimension of the corresponding eigenspace $Z(T-\lambda I)$ is called the <u>geometric multiplicity of</u> λ .

If the algebraic multiplicity of λ is 1 , then λ is called a <u>simple</u> eigenvalue of T . If λ is a pole of $R(z)$ of order 1 , (i.e., $D_\lambda = 0$) , then λ is said to be a <u>semisimple</u> eigenvalue of T .

Note that an isolated point λ of $\sigma(T)$ is a semisimple eigenvalue of T if and only if $P_\lambda(X) = Z(T-\lambda I)$ (by Lemma 7.1(b)), i.e., the corresponding spectral subspace coincides with the eigenspace.

PROPOSITION 7.6 Let λ be a pole of $R(z)$. (This condition is satisfied if λ is an eigenvalue of T of finite algebraic multiplicity.)

(a) λ is a semisimple eigenvalue of T if and only if $(T-\lambda I)x$ is not an eigenvector of T corresponding to λ for any $x \in X$.

(b) λ is simple if and only if there is a unique (up to scalar multiples) eigenvector φ of T corresponding to λ , and there is no $x \in X$ such that $(T-\lambda)x = \varphi$.

Proof Let ℓ be the order of the pole of $R(z)$ at λ . Then by (ii) of Lemma 7.1(b),

$$R(P_\lambda) = Z((T-\lambda I)^\ell) .$$

(a) By Proposition 7.3, we see that $\ell = 1$ if and only if

$$Z((T-\lambda I)^2) = Z(T-\lambda I) .$$

Clearly, this happens if and only if there is no $x \in X$ with $(T-\lambda I)x \neq 0$, but $(T-\lambda I)[(T-\lambda I)x] = 0$, i.e., $(T-\lambda I)x$ is not an eigenvector of T corresponding to λ for any $x \in X$.

(b) By (7.15), λ is simple if and only $\ell = 1$ and the geometric multiplicity g of λ is 1. Hence the desired result follows by part (a). //

When the geometric multiplicity of λ is greater than 1, it is possible that for a basis $\varphi_1,\ldots\varphi_g$ of the eigenspace $Z(T-\lambda I)$, each of the equations $(T-\lambda I)x = \varphi_i$, $i = 1,\ldots,g$, has no solution in X, but $(T-\lambda I)x = \varphi$ does have a solution for some $0 \neq \varphi \in Z(T-\lambda I)$: Let $X = \mathbb{C}^3$, and $T[x(1),x(2),x(3)]^t = [\lambda x(1)+x(2), \lambda x(2), \lambda x(3)]^t$. Then $\varphi_1 = [1,0,1]^t$ and $\varphi_2 = [1,0,-1]^t$ constitute a basis of $Z(T-\lambda I)$, but none of equations $(T-\lambda I)x = \varphi_i$, $i = 1,2$, has a solution in X. However, if we let $\varphi = [1,0,0]^t$, then the equation $(T-\lambda I)x = \varphi$ has $[x(1),1,x(3)]^t$ as a solution for all $x(1)$ and $x(3)$ in $\mathbb{C}$. (In particular, λ is not a semisimple eigenvalue of T.)

Remark 7.7 The term 'geometric multiplicity' is self-explanatory, since it is the dimension of the corresponding eigenspace. To explain the term 'algebraic multiplicity' we proceed as follows.

Let the algebraic multiplicity of λ be $m < \infty$. Then λ is a pole of $R(z)$. Let ℓ be order of this pole. Since D_λ is quasi-nilpotent, and since $P_\lambda(X)$ has dimension $m < \infty$, we see by Proposition 5.6 that $D_\lambda|_{P_\lambda(X)}$ is, in fact, nilpotent, and ℓ is the smallest positive integer such that $(D_\lambda|_{P_\lambda(X)})^\ell = 0$. Considering the representation (7.5)

$$T|_{P_\lambda(X)} = \lambda I|_{P_\lambda(X)} + D_\lambda|_{P_\lambda(X)} ,$$

we see that $T|_{P_\lambda(X)}$ is represented, with respect to a suitable basis of $P_\lambda(X)$, in the Jordan canonical form (cf. (5.14)) by the $m \times m$ matrix

$$M = \begin{bmatrix} \lambda & \delta_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \delta_m \\ 0 & & & \lambda \end{bmatrix} ,$$

where each δ_j is either 0 or 1, $2 \leq j \leq m$. Thus, λ is a root of order m of the characteristic polynomial of M, and hence the algebraic multiplicity of λ is said to be m.

By looking at the δ_j's in the above representation, one can also determine the geometric multiplicity g of λ and the order ℓ of the pole at λ. Let $x = [x(1),\ldots,x(m)]^t \in \mathbb{C}^m$. Then

$$Mx = \lambda x + [\delta_2 x(2),\ldots,\delta_m x(m),0]^t .$$

Thus, x is an eigenvector corresponding to λ if and only if $\delta_j x(j) = 0$ for each $j = 2,\ldots,m$. Hence $[1,0,\ldots,0]^t$ is an eigenvector, and if $\delta_j = 0$ for some j, then $[0,\ldots,0,1,0,\ldots,0]^t$ is also an eigenvector, where 1 occurs in the j-th place; these vectors form a basis of the eigenspace corresponding to λ. Thus, the geometric multiplicity g of λ equals one plus the number of zeros among $\delta_2,\ldots,\delta_m$. Also, it can be seen that if k is the maximum number of consecutive 1's among $\delta_2,\ldots,\delta_m$, then the $(k+1)$-st power of the matrix

$$\begin{bmatrix} 0 & \delta_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \delta_m \\ 0 & & & 0 \end{bmatrix}$$

equals the zero matrix, and no smaller power does so. Thus, the order ℓ of the pole at λ equals one plus the maximum number of consecutive 1's among $\delta_2, \ldots, \delta_m$. Notice that in the notation used in the description of the Jordan canonical form of a nilpotent operator in Section 5, we have $g = p_\ell$, while m and ℓ have the same meanings as used in this section.

We give some simple examples to illustrate the above considerations. Let $m = 4$ and let $T|_{P_\lambda}$ be represented by one of the following Jordan canonical forms:

$$M_1 = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad M_2 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad M_3 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$M_4 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad M_5 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For M_1, $g = 4$ and $\ell = 1$; for M_2, $g = 3$ and $\ell = 2$; for M_3, $g = 2$ and $\ell = 2$; for M_4, $g = 2$ and $\ell = 3$ and for M_5, $g = 1$ and $\ell = 4$. Note that these are the only possibilities for the case $m = 4$.

We say that λ is a <u>discrete spectral value</u> of T if λ is an isolated point of $\sigma(T)$ and the corresponding spectral projection P_λ has finite rank, i.e., λ is an eigenvalue of T of finite algebraic multiplicity. The set of all discrete spectral values of T constitutes the <u>discrete spectrum</u> $\sigma_d(T)$ <u>of</u> T. The discrete spectral values of T form by far the most tractable part of $\sigma(T)$, as we shall see in the later sections. See Corollary 3 of Appendix I for a characterization of $\sigma_d(T)$.

In order to tell when the spectral projection P_Γ associated with a curve Γ in $\rho(T)$ has finite rank, we prove a preliminary result.

LEMMA 7.8 Let a curve Γ in $\rho(T)$ enclose only a finite number of (isolated) points $\lambda_1,\ldots,\lambda_n$ of $\sigma(T)$. If P_j denotes the spectral projection associated with λ_j, $1 \le j \le n$, and $P = P_\Gamma$, then

$$P = P_1 + \ldots + P_n ,$$
$$P_jP_k = 0 , \qquad j \neq k ,$$
$$R(P) = R(P_1) \oplus \ldots \oplus R(P_n) ,$$
$$TP = \sum_{j=1}^{n} (\lambda_jP_j+D_j) ,$$

where D_j is the quasinilpotent operator $(T-\lambda_jI)P_j$.

Proof For each j, let Γ_j be a curve such that $\lambda_j \in \mathrm{Int}\,\Gamma_j$ and $\Gamma_j \subset \mathrm{Int}\,\Gamma \cap \mathrm{Ext}\,\Gamma_k$, $k \neq j$.

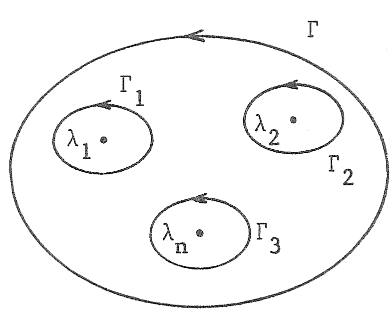

Figure 7.1

Then by Cauchy's theorem (Theorem 4.3(a)),

$$\int_\Gamma R(z)dz - \int_{\Gamma_1} R(z)dz - \ldots - \int_{\Gamma_n} R(z)dz = 0 ,$$

so that $P = P_1 + \ldots + P_n$. Also, if $j \neq k$, then by (4.17),

$$\begin{aligned} P_jP_k &= \frac{-1}{2\pi i} \int_{\Gamma_k} P_jR(z)dz \\ &= \frac{-1}{2\pi i} \int_{\Gamma_k} \left[\frac{-1}{2\pi i} \int_{\Gamma_j} R(w)R(z)dw\right]dz \ . \\ &= \left[\frac{-1}{2\pi i}\right]^2 \int_{\Gamma_k} \left[\int_{\Gamma_j} \frac{R(w)-R(z)}{w - z}\, dw\right]dz \ . \end{aligned}$$

But, for all $w \in \Gamma_j$ and $z \in \Gamma_k$, we have

$$\int_{\Gamma_k} \frac{dz}{w - z} = \int_{\Gamma_j} \frac{dw}{w - z} = 0 \ ,$$

since $w \in \operatorname{Ext} \Gamma_k$ and $z \in \operatorname{Ext} \Gamma_j$. Hence for all $j \neq k$, we have $P_jP_k = 0$, so that $R(P_j) \cap R(P_k) = \{0\}$. This shows that $R(P) = R(P_1) \oplus \dots \oplus R(P_n)$. Finally, since $D_j = TP_j - \lambda_jP_j$, we have

$$\begin{aligned} TP &= TP_1 + \dots + TP_n \\ &= \sum_{j=1}^{n} (\lambda_jP_j+D_j) \ . \qquad // \end{aligned}$$

THEOREM 7.9 Let Γ be a curve in $\rho(T)$. Then the associated spectral projection P_Γ is of finite rank if and only if $\sigma(T) \cap \operatorname{Int} \Gamma$ consists of a finite number of discrete spectral values of T, and in that case, the rank of P_Γ equals the sum of the algebraic multiplicities of the eigenvalues of T inside Γ.

Proof Let $Y = P_\Gamma(X)$ and $\dim Y < \infty$. Then T_Y is a finite dimensional operator and hence $\sigma(T_Y)$ consists of a finite number of (isolated) eigenvalues $\lambda_1,\dots,\lambda_n$ of $T|_Y$. But by (6.10) (the spectral decomposition theorem),

$$\sigma(T_Y) = \sigma(T) \cap \operatorname{Int} \Gamma \ .$$

Hence $\lambda_1,\dots,\lambda_n$ are isolated points of $\sigma(T)$. If P_j denotes the spectral projection associated with λ_j, then by Lemma 7.8,

$$R(P_\Gamma) = R(P_1) \oplus \ldots \oplus R(P_n) .$$

Hence each P_j has finite rank, i.e., $\lambda_j \in \sigma_d(T)$.

Conversely, let

$$\sigma(T) \cap \operatorname{Int} \Gamma = \{\lambda_1, \ldots, \lambda_n\} ,$$

where each $\lambda_j \in \sigma_d(T)$. Then again by Lemma 7.8,

$$\dim P_\Gamma = \sum_{j=1}^{n} \dim P_j < \infty .$$

Note that $\dim P_j$ is the algebraic multiplicity of λ_j . //

We now describe some general situations where discrete spectral values are always encountered.

(i) Let X be finite dimensional, and $T \in BL(X)$. Then $\sigma(T) = \sigma_d(T) = \{\lambda_1, \ldots, \lambda_n\}$, say. If $\Gamma \subset \rho(T)$ encloses all the spectral values of T , then $P_\Gamma = I = P_1 + \ldots + P_n$, and by Lemma 7.8, we have

$$T = \sum_{j=1}^{n} (\lambda_j P_j + D_j) \tag{7.16}$$

where each D_j is nilpotent. Let $T_j = (\lambda_j P_j + D_j)|_{R(P_j)}$. We have seen earlier that in a suitable basis for $R(P_j)$, T_j is represented by the matrix

$$J_j = \begin{bmatrix} \lambda_j & \delta & 0 & \cdots & 0 \\ 0 & & \ddots & \delta & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & \delta \\ 0 & \cdots & & 0 & \lambda_j \end{bmatrix}$$

where δ denotes either 0 or 1 . Thus, we obtain a block diagonal matrix representation

$$J = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & J_n \end{bmatrix}$$

of T, known as a <u>Jordan canonical form</u>. It is immediate from the above representation that

$$\text{(7.17)} \qquad \begin{aligned} \det(J-zI) &= \prod_{j=1}^{n} (\lambda_j - z)^{m_j} \\ \operatorname{tr}(T) = \operatorname{tr}(J) &= \sum_{j=1}^{n} m_j \lambda_j , \end{aligned}$$

where m_j is the algebraic multiplicity of λ_j.

In this case, the range of the spectral projection P_j associated with T and λ_j is the generalized eigenspace $\{x \in X : (T-\lambda_j I)^{m_j} = 0\}$ of T corresponding to λ_j, and its null space is the direct sum of the remaining generalized eigenspaces of T:

$$\text{(7.18)} \qquad \begin{aligned} Z(P_j) &= Z\Big[I - \sum_{i=1, i\neq j}^{n} P_i\Big] = R\Big[\sum_{i=1, i\neq j}^{n} P_i\Big] \\ &= \bigoplus_{i=1, i\neq j}^{n} R(P_i) = \bigoplus_{i=1, i\neq j}^{n} \Big\{x \in X : (T-\lambda_i I)^{m_i} x = 0\Big\} . \end{aligned}$$

(ii) Let $T \in BL(X)$ be a compact operator, i.e., let the closure of the set $\{Tx : x \in X , \|x\| \leq 1\}$ be compact in X. Then one shows that $T - I$ is one to one if and only if it is onto. ([L], 18.4(b)). This implies that every nonzero spectral value of T is, in fact, an eigenvalue of T. The compactness of T then implies that the set of eigenvalues of T is countable, and has no limit point except possibly the number 0 ([L], 18.2). Thus, every nonzero λ in $\sigma(T)$ is an isolated point of $\sigma(T)$. Let $\Gamma \subset \rho(T)$ separate λ from the rest of $\sigma(T)$ *and also from zero*. Then

$$
\begin{aligned}
P_\lambda = P_\Gamma &= -\frac{1}{2\pi i}\int_\Gamma R(z)dz \\
&= -\frac{1}{2\pi i}\int_\Gamma \left[\frac{I}{z} + R(z)\right]dz \\
&= -\frac{1}{2\pi i}\int_\Gamma \frac{1}{z}\,[I + zR(z)]dz \\
&= -\frac{1}{2\pi i}\int_\Gamma \frac{1}{z}\,TR(z)dz \ ,
\end{aligned} \tag{7.19}
$$

by (5.4). Now, since T is compact, so is $TR(z)/z$ for every $z \in \Gamma$. Hence P_Γ is compact, being the limit (in BL(X)) of the Riemann-Stieltjes sums (4.5) of compact operators. But a compact projection must have finite rank by Corollary 3.9, so that the rank of P_λ is finite.

Thus, every nonzero spectral value of a compact operator is an eigenvalue of finite algebraic multiplicity, i.e., it is a discrete spectral value. If $0 \in \sigma_d(T)$ also, then $\sigma(T)$ will consist of a finite number of discrete spectral values, and Lemma 7.8 will imply that X is finite dimensional. Hence whenever X is infinite dimensional and T is compact, we have

$$\sigma_d(T) = \sigma(T) \setminus \{0\} \ .$$

Let, now, $\lambda_1, \lambda_2, \ldots$ denote the nonzero (isolated) spectral values of T. Let P_j denote the spectral projection (of finite rank) associated with λ_j, $j = 1,2,\ldots$. If we let

$$Q_n = P_1 + \ldots + P_n \ , \quad n = 1,2,\ldots \ ,$$

then we have as in Lemma 7.8,

$$TQ_n = \sum_{j=1}^{n} (\lambda_j P_j + D_j)$$

where each D_j is nilpotent. However, T need not have the infinite representation

$$\sum_{j=1}^{\infty} (\lambda_j P_j + D_j) \ ,$$

as the example of the Volterra integration operator V shows. In this case, we have $\sigma(V) = \{0\}$, so that there is no nonzero spectral point of V, but at the same time $V \neq 0$. In the next section we shall consider compact *normal* operators on a Hilbert space for which the above infinite expansion is valid.

Examples of isolated spectral values.

(i) Let $X = \ell^2$, and let $T \in BL(X)$ be represented by the diagonal infinite matrix

$$\mathrm{diag}(1, \tfrac{1}{2}, 1, \tfrac{1}{3}, 1, \tfrac{1}{4}, \ \ldots \ ) \ .$$

Then for $z \neq 0 \ , 1 \ , \frac{1}{2} \ , \frac{1}{3} \ , \ \ldots \ ,$ the resolvent operator $R(z)$ is represented by the matrix

$$\mathrm{diag}(\frac{1}{1-z} \ , \ \frac{2}{1-2z} \ , \ \frac{1}{1-z} \ , \ \frac{3}{1-3z} \ , \ \ldots \ ) \ .$$

The eigenvalue $\lambda = 1$ has infinite geometric and algebraic multiplicities since each e_{2n+1}, $n = 0,1,2,\ldots,$ is an eigenvector of T corresponding to $\lambda = 1$; the associated spectral projection P_1 is given by termwise integration of $R(z)$ over $\Gamma(t) = 1 + re^{it}$, $0 \leq t \leq 2\pi$, $0 < r < 1/2$, it is represented by

$$\mathrm{diag}(1, \ 0, \ 1, \ 0, \ \ldots \ ) \ .$$

Hence it follows that

$$D_1 = (T-I)P_1 = 0 \ .$$

Thus, $\ell = 1$, i.e., λ is a semisimple (but not a simple) eigenvalue of T. In this case, $S_1 = \lim_{z \to 1} R(z)(I-P_1)$ is represented by

$$\lim_{z\to 1} \mathrm{diag}\left(\frac{1}{1-z}, \frac{2}{1-2z}, \ldots\right)\mathrm{diag}(0, 1, 0, 1, \ldots)$$
$$= \mathrm{diag}\left(0, -2, 0, \frac{-3}{2}, 0, \frac{-4}{3}, \ldots\right) .$$

(ii) Let $X = \ell^2(\mathbb{Z})$, the space of all square summable doubly infinite complex sequences. Let $T \in BL(X)$ be given by the matrix

$$\left[\begin{array}{ccc:ccccc} \ddots & & & & & & & \\ & 1/3 & & & & & & \\ & & 1/2 & & & & & \\ \hdashline & & & 1 & 1 & & & \\ & & & & 1 & 1/2 & & \\ & & & & & 1 & 1/3 & \\ & & & & & & 1 & \ddots \\ & & & & & & & \ddots \end{array}\right] ,$$

where all the remaining entries are equal to zero. It can be verified ([L], Problem 18(ii)) that

$$\sigma(T) = \{0\} \cup \{1/n : n = 1,2,\ldots\} ,$$

that $\lambda = 1$ has geometric multiplicty 1, but infinite algebaic multiplicty; it is, in fact, an essential singularity of $R(z)$.

(iii) Let $X = L^2([0,1])$ and

$$Tx(s) = \int_0^1 k(s,t)x(t)dt , \quad x \in X , \quad s \in [0,1] ,$$
$$k(s,t) = \begin{cases} s/2 , & 0 \le s < t \\ (2t-s)/2 , & t \le s < 1 \end{cases} .$$

Then it can be checked that $Tx = y$, $x \in X$ if and only if y' is absolutely continuous on $[0,1]$, $y'' \in X$ and

$$-y'' = x , \quad y(0) = 0 , \quad y'(0) + y'(1) = 0 .$$

The eigenvalues of the compact operator T are $1/[(2j-1)\pi]^2$, $j = 1,2,\ldots$. We can verify that corresponding to the eigenvalue

$\lambda = 1/\pi^2$, $x_1(t) = \sin \pi t$ is an eigenvector, while $x_2(t) = t \cos \pi t$ is a generalized eigenvector. In fact, in this case $g = 1$, $m = 2 = \ell$. Thus, λ is not a semisimple eigenvalue.

(iv) Let $X = L^2([-1,1])$ and

$$Tx(s) = \int_{-1}^{1} k(s,t)x(t)dt \ , \ x \in X \ , \ s \in [-1,1] \ ,$$

$$k(s,t) = \frac{\sqrt{e}}{e-1} \begin{cases} e^{(1+t-s)/2} + e^{(-1-t+s)/2} \ , & -1 \le t < s \\ \\ e^{(-1+t-s)/2} + e^{(1-t+s)/2} \ , & s \le t < 1 \end{cases} .$$

Then $Tx = y$, $x \in X$ if and only if y' is absolutely continuous on $[-1,1]$, $y'' \in X$, and

$$-y'' + \frac{1}{4} y = x \ , \ y(-1) = y(1) \ , \ y'(-1) = y'(1) \ .$$

The eigenvalues of T are $4/(4\pi^2 n^2 + 1)$, $n = 0,1,2,\ldots$. Corresponding to the eigenvalue $\lambda = 4$, we have only one linearly independent eigenfunction $x_0(t) = 1$. But corresponding to the eigenvalue $\lambda_n = 4/(4\pi^2 n^2 + 1)$, $n = 1,2,\ldots,$ we have the eigenfunctions $x_{n,1}(t) = \sin n\pi t$ and $x_{n,2}(t) = \cos n\pi t$; in fact, in this case $g = m = 2$, $\ell = 1$.

(v) The nonzero eigenvalues of many operators which describe various physical situations are *simple*. We now quote some general results regarding the 'simplicity' of eigenvalues.

Let X be finite dimensional and $T \in BL(X)$ be representd by a matrix $K = (k_{i,j})$. *Perron's theorem* states that if $k_{i,j} > 0$ for all i,j , then T has a positive simple eigenvalue which exceeds the moduli of all other eigenvalues. *Frobenius' generalization* of this theorem says that if $k_{i,j} \ge 0$ for all i,j and K is irreducible

(i.e., there is no permutation matrix P such that
$P^H KP = \begin{bmatrix} K_{1,1} & K_{1,2} \\ 0 & K_{2,2} \end{bmatrix}$, where $K_{1,1}$ and $K_{2,2}$ are square matrices of order less than the order of K), then all the eigenvalues of T of largest modulus are simple. ([G], p.53). Another fundamental result states that if (a) $k_{i,j} \geq 0$ for all i and j, (b) all the minors of K have nonnegative determinants, (c) $k_{i,j} > 0$ whenever $|i-j| \leq 1$, and (d) $\det K > 0$, then all the eigenvalues of T are positive and simple. ([G], p.105).

Here are some infinite dimensional analogues of some of the above results. Let $X = \ell^2$, and a compact normal operator T be represented by the infinite matrix $(k_{i,j})$. If $k_{i,j} \geq 0$ for all i,j and $k_{i,j} > 0$ whenever $|i-j| \leq 1$, then $\|T\|$ is a simple eigenvalue of T ([KR], Prop. (β''), Sec.3). Similarly, let $X = L^2([a,b])$ and let T be a compact normal integral operator

$$Tx(s) = \int_a^b k(s,t)x(t)dt \ , \ x \in X \ , \ s \in [a,b] \ ,$$

where the kernel k is continuous on $[a,b]\times[a,b]$, $k(s,t) \geq 0$ for all s,t, and $k(t,t) > 0$ for all t. Then $\|T\|$ is a simple eigenvalue of T ([KR], Prop.(β'), Sec.3).

Problems

7.1 Let $X = \ell^2$ and

$$T[x(1), x(2), x(3), \ldots]^t = [x(2), \frac{x(3)}{2}, \frac{x(4)}{3}, \ldots]^t .$$

Then T is quasi-nilpotent but not nilpotent. $R(z)$ has an essential singularity at 0.

7.2 Let λ be an isolated point of $\sigma(T)$. If $x \in R(P_\lambda)$ and $z \in \rho(T)$, then $R(z)x = -\sum_{k=0}^{\infty} (T-\lambda I)^k x \,/\, (z-\lambda)^{k+1}$. Also,

$$R(P_\lambda) = \{x \in X : \|(T-\lambda I)^n x\|^{1/n} \to 0 \text{ as } n \to \infty\} .$$

7.3 Let $\dim P_\lambda(X) = 3$ and assume that there are two linearly independent eigenvectors corresponding to λ , but no more. Then $TP_\lambda \neq \lambda P_\lambda$, but $T^2 P_\lambda = \lambda P_\lambda(2T-\lambda I)$.

7.4 Let λ be an isolated point of $\sigma(T)$. Then the function $z \mapsto R(z)(I-P_\lambda)$ has a removable singularity at λ . If $\Gamma \subset \rho(T)$ and $\operatorname{Int} \Gamma$ contains only a finite number of points of $\sigma(T)$, then the function $z \mapsto R(z)(I-P_\Gamma)$ has only removable singularities in $\operatorname{Int} \Gamma$.

7.5 Let $\lambda \in \sigma_d(T)$ and $Y = R(P_\lambda)$. Then for $n = 1,2,\ldots$,

$$R((T-\lambda I)^n) = \{y \in X : P_\lambda y \in R((T_Y-\lambda I_Y)^n)\} ,$$

and it is a closed subspace of X .

7.6 Let λ be a pole of $R(z)$ of order ℓ . Then every nonzero element of $R(D_\lambda^{\ell-1})$ is an eigenvector of T corresponding to λ (Note: $D_\lambda^0 = P_\lambda$) .

7.7 Let $A , B \in BL(X)$. Then $\sigma_d(AB) \setminus \{0\} = \sigma_d(BA) \setminus \{0\}$. Let $0 \neq \lambda \in \sigma_d(AB)$ have algebraic (resp., geometric) multiplicity m (resp., g), and let λ be a pole of order ℓ of $R(AB,z)$. Then the same holds if we replace AB by BA . (Cf. Problem 5.1.) In fact, $AP_\lambda(BA) = P_\lambda(AB)A$. If X is finite dimensional, then 0 is an eigenvalue of the same algebraic multiplicity of AB and of BA , and it is a pole of the same order of $R(AB,z)$ and $R(BA,z)$, but the dimensions of $Z(AB)$ and $Z(BA)$ may not be equal.

7.8 Let $z_0 \in \rho(T)$. A complex number λ is an isolated point of $\sigma(T)$ if and only if $1/(\lambda - z_0)$ is an isolated point of $\sigma(R(z_0))$; in that case, the associated spectral projections are the same and $Z(T-\lambda I) = Z(R(z_0)-I/(\lambda-z_0))$. Moreover, the order of the pole of $R(T,z)$ at λ is the same as the order of the pole of $R(R(z_0),z)$ at $1/(\lambda-z_0)$. (Hint: (5.2) and Problem 4.8)

7.9 Let λ be an isolated point of $\sigma(T)$. For $z \in \rho(T)$, we have

$$\left[S_\lambda - \frac{I}{z-\lambda}\right]^{-1} = -(z-\lambda)I - (z-\lambda)^2 R(z)(I-P_\lambda) .$$

Then $\mu(\neq \lambda)$ is an isolated point of $\sigma(T)$ if and only if $1/(\mu-\lambda)$ is an isolated point of $\sigma(S_\lambda)$; in that case, the associated spectral projections are the same, and $Z(T-\mu I) = Z(S_\lambda - I/(\mu-\lambda))$.

7.10 Let λ be a pole of $R(z)$ of order ℓ , $A = T - \lambda I$ and $S = S_\lambda$. Then A and S satisfy $SAS = S$, $A^\ell SA = A^\ell$, $SA = AS$ (i.e., S_λ is the Drazin inverse of $T - \lambda I$) . If λ is semisimple, then $SAS = S$, $ASA = A$, $SA = AS$ (i.e., S_λ is the group inverse of $T - \lambda I$) . Let X be a Hilbert space and λ semisimple. Then the projection P_λ is orthogonal if and only if $SA = A^* S^*$ (i.e., S is the Moore-Penrose inverse of $T - \lambda I$; see the Penrose conditions on page 403).

7.11 Let $X = L^2([-\pi,\pi])$, and for $x \in X$,

$$Tx(s) = \int_{-\pi}^{\pi} k(s,t)x(t)dt , \quad s \in [-\pi,\pi] ,$$

$$k(s,t) = \frac{1}{2\sqrt{2}} \begin{cases} \sin \sqrt{2}(s-t) + (\cot \pi\sqrt{2}) \cos \sqrt{2}(s-t) , & -\pi \le t \le s \le \pi \\ \sin \sqrt{2}(t-s) + (\cot \pi\sqrt{2}) \cos \sqrt{2}(t-s) , & -\pi \le s \le t \le \pi . \end{cases}$$

The eigenvalues of T are $1/(2-n^2)$, $n = 0,1,\ldots$. The dominant eigenvalue 1 is semisimple but not simple.