

WEAKLY INTEGRABLE SEMIGROUPS

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The connection between Markov processes and semigroups of continuous linear operators is now well established. However, early in the development of the theory William Feller observed that the use of strongly continuous semigroups is not always appropriate [3].

Subsequently, the interests of probabilists turned towards other directions and to new methods. As a consequence, there still seems to be no coherent account of the class of semigroups most appropriate for the study of Markov processes and diffusion processes. The work of E.B. Dynkin [2] is suggestive in this context, but it is not sufficiently developed. The approach suggested here is based on a number of principles in common with many problems in analysis.

Weak (Pettis-type) integration, as opposed to the strong (Bochner) integrals used in the standard theory is the natural tool for this situation. Secondly, it is often necessary to weaken the topology of the underlying vector space, even to utilize the weak integral - a feature in common with the spectral theory of operators [5]. Another aspect of the present approach is that the "infinitesimal generator" of the semigroup is defined directly in terms of the resolvent, instead of by differentiation as is most commonly done; this has the advantage of substituting integral operators and equations for differential ones - a time-honoured and successful method in analysis [1;p.207].

Characteristic examples of the type of semigroups treated here arise from diffusion processes generated by an elliptic differential operator L

in \mathbb{R}^d . The operator L has the form

$$L\phi = \sum_{i,j=1}^d a_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial \phi}{\partial x_i}, \quad \phi \in C_0^\infty(\mathbb{R}^d).$$

The coefficients a_{ij}, b_i are assumed to be locally bounded, Borel measurable functions on \mathbb{R}^d for $i, j=1, \dots, d$, and L satisfies the ellipticity condition

$$\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq 0, \quad x \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^d.$$

The terminology is not fixed among authors, but the following will do.

The definition of a Markov process is given in [1], [2] and [6].

DEFINITION A continuous Markov process $(\Omega, \mathcal{S}, (P^x)_{x \in \mathbb{R}^d}; (X_t)_{t > 0})$ is said to be a *diffusion associated with* L if

$$P^x(\phi \circ X_t) = \phi(x) + \int_0^t P^x(L\phi \circ X_s) ds$$

for every $\phi \in C_0^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $t > 0$.

Existence and uniqueness results for diffusions associated with L are given in [2] by exploiting methods from partial differential equations, and from a more probabilistic direction in [6]. It is clear that the regularity theory of parabolic equations is largely superfluous in this context, so the conditions on the coefficients of L imposed by that theory are not the natural ones; the treatment in [6] is therefore more appealing.

Denote the bounded Borel measurable functions on \mathbb{R}^d with the sup-norm by $L^\infty(\mathbb{R}^d)$. Given a diffusion $(\Omega, \mathcal{S}, (P^x)_{x \in \mathbb{R}^d}; (X_t)_{t > 0})$ associated with L , a family $(T_t)_{t > 0}$ of continuous linear operators on $L^\infty(\mathbb{R}^d)$

may be defined by the relation

$$T_t f(x) = P^x(f \circ X_t) ,$$

for each $f \in L^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, and $t > 0$.

The family $(T_t)_{t>0}$ has the semigroup property

$$T_{t+s} = T_t T_s , \quad t, s > 0 ,$$

and satisfies the following conditions.

D_1 : $\|T_t\| \leq 1$ and $T_t 1 = 1$ for all $t > 0$.

D_2 : If $f \in L^\infty(\mathbb{R}^d)$ and $f \geq 0$, then $T_t f \geq 0$ for all $t > 0$.

D_3 : Let $M(\mathbb{R}^d)$ be the signed Borel measures on \mathbb{R}^d , put in duality with $L^\infty(\mathbb{R}^d)$ by the pairing

$$\langle f, \mu \rangle = \int_{\mathbb{R}^d} f \, d\mu , \quad f \in L^\infty(\mathbb{R}^d) , \quad \mu \in M(\mathbb{R}^d) .$$

Then for each $t > 0$, T_t is $\sigma(L^\infty(\mathbb{R}^d), M(\mathbb{R}^d))$ -continuous.

D_4 : For each $\phi \in C_0^\infty(\mathbb{R}^d)$, $\mu \in M(\mathbb{R}^d)$, $t > 0$,

$$\langle T_t \phi, \mu \rangle = \langle \phi, \mu \rangle + \int_0^t \langle T_s L\phi, \mu \rangle \, ds .$$

Condition D_3 means that each of the operators T_t arises from a transition function, and D_4 follows from Fubini's theorem. A Markov process can always be constructed from a semigroup satisfying $D_1 - D_3$, but

it need not be a continuous process.

The abstract setting is as follows. Let E be a locally convex space. The space of continuous linear operators on E is denoted by $L(E)$. Let F be a separating subspace of the continuous dual E' of E .

DEFINITION A semigroup $S : \mathbb{R}^+ \rightarrow L(E)$ of continuous linear operators on E is said to be an F -semigroup if the following conditions hold.

S_1 : Let S' be the adjoint semigroup on E' defined by

$$\langle x, S'(t)\zeta \rangle = \langle S(t)x, \zeta \rangle, \quad x \in E, \quad \zeta \in E', \quad t > 0.$$

Then F has an S' -invariant subspace which separates E .

S_2 : There exists $\omega_0 \geq 0$ such that for each $x \in E$ and $\mu > \omega_0$, the function

$$t \rightarrow e^{-\mu t} \langle S(t)x, \xi \rangle, \quad t > 0$$

is integrable on \mathbb{R}^+ for each $\xi \in F$, and there exists

$R(\mu)x \in E$ such that

$$\langle R(\mu)x, \xi \rangle = \int_0^{\infty} e^{-\mu t} \langle S(t)x, \xi \rangle dt, \quad \xi \in F.$$

When $E = L^\infty(\mathbb{R}^d)$ and $F = M(\mathbb{R}^d)$, property D_3 ensures that S_1 holds, and Fubini's theorem gives S_2 (the *measurability* of the process is important here), so $(T_t)_{t>0}$ is surely an F -semigroup on E .

By Fubini's theorem, we have the following relation among the operators

$R(\mu) : E \rightarrow E$, $\mu > \omega_0$ defined above.

PROPOSITION For every $\mu, \nu > \omega_0$

$$R(\mu) - R(\nu) = (\nu - \mu)R(\mu)R(\nu) .$$

The *resolvent relation* above allows us to define the *infinitesimal generator* $A : D(A) \rightarrow E$ of S by $A = \mu - R(\mu)^{-1}$ whenever one (and hence all) of the linear maps $R(\mu)$ is injective. In the remainder we assume that A is so defined.

THEOREM [4] Let S be an F -semigroup on E with infinitesimal generator A . Then $S(t)D(A) \subset D(A)$ and $S(t)A = A S(t)$ for each $t > 0$.

For every $x \in D(A)$,

$$\langle S(t)x, \xi \rangle = \langle x, \xi \rangle + \int_0^t \langle A S(s)x, \xi \rangle ds, \quad \xi \in F, t > 0 .$$

This integral equation means that A is a type of weak derivative of S . For the semigroup generated by a diffusion associated with L , the relation is an extension of property D_4 whenever A is an extension of L .

The following result is an analogue of the Hille-Yosida-Phillips theorem for strongly continuous semigroups.

THEOREM [4] Let E be a locally convex space with the closed graph property. Let F be a separating subspace of E' .

Let $R(\mu)$, $\mu > \omega_0$ be a resolvent family of $\sigma(E, F)$ -continuous linear maps on E such that for every $x \in E$, $\mu R(\mu)x \rightarrow x$ in the topology of uniform convergence on $\sigma(E', E)$ -compact subsets of E' as $\mu \rightarrow \infty$.

Then there exists a $\sigma(E, F)$ -closed, closed and densely defined linear map $A : D(A) \rightarrow E$ such that

$$R(\mu) = (\mu - A)^{-1}, \quad \mu > \omega_0.$$

Let $H = D(A)$, $K = R(\mu)'F'$. The following conditions (a) and (b) are equivalent.

(a) There exists an F -semigroup S on E such that for all $x \in E$, the set $\{e^{-\omega_0 t} S(t)x : t > 0\}$ is $\sigma(E, F)$ -bounded in E , and for all $\mu > \omega_0$,

$$\langle R(\mu)x, \xi \rangle = \int_0^{\infty} e^{-\mu t} \langle S(t)x, \xi \rangle dt, \quad \xi \in F'.$$

(b) (i) For each $x \in E$, the maps

$$(\mu_1, \dots, \mu_n) \rightarrow [(\mu_1 - \omega_0)R(\mu_1)] \dots [(\mu_n - \omega_0)R(\mu_n)]x,$$

$$\mu_1, \dots, \mu_n > \omega_0, \quad n \in \mathbb{N}$$

are $\sigma(E, F)$ -continuous in E and uniformly $\sigma(E, F)$ -bounded.

(ii) For each $x \in E$, $\omega > \omega_0$, the set

$$\{(\mu - \omega)^n R(\mu)^n x : \mu \geq \omega, n=1, 2, \dots\}$$

is a relatively $\sigma(E, K)$ -compact subset of E .

(iii) For each $\xi \in F$ (and for a separating set of $\xi \in F$, respectively), $\omega > \omega_0$, the set

$$\{(\mu - \omega)^n [R(\mu)^n]'\xi : \mu > \omega, n=1, 2, \dots\}$$

is a relatively $\sigma(E', H)$ -compact subset of E' (a relatively $\sigma(F, H)$ -compact subset of F , respectively).

The operators $S(t)$, $t > 0$ are $\sigma(E,F)$ -continuous if and only if, in addition, the set in condition (b) (iii) is a relatively $\sigma(F,H)$ -compact subset of F . If S is $\sigma(E,F)$ -continuous at $t = 0$, then the sets K, H can be replaced by F, E in (b) (ii) and (iii) respectively.

An apparent advantage of the semigroup approach to diffusion processes is that if there is not a unique diffusion associated with L , then there is some insight into the "boundary conditions" which can be placed on the infinitesimal generator to obtain uniqueness; at least this is the case in one dimension. For higher dimensions it remains to be seen whether there is any real advantage to the method.

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