NONCONFORMING ELEMENTS, PATCH TESTS AND
INTER-ELEMENT CONSTRAINTS

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1. INTRODUCTION

For a variational problem requiring that the admissible functions have partial derivatives of order m that are square integrable, it is sufficient that the piecewise polynomials in the finite element basis have continuous derivatives up to order m − 1 . That is, $v_h \in C^{m-1}$ where h denotes the mesh parameter for a finite element discretization of the domain. Hence for second order problems (2m = 2) the basis is C^0 (globally continuous) while for fourth-order problems (2m = 4) it is C^1 (derivatives to first order are continuous). Such finite element bases are said to be conforming. If $v_h \notin C^{m-1}$ the basis is said to be nonconforming.

Because of their simplicity and relatively low degree, nonconforming finite elements were first applied to finite element calculations of plate bending problems in the mid 1960's. These exploratory studies produced conflicting results − in some instances they led to accurate converging solutions, while in other calculations the method failed to converge. In particular, for certain test problems with nonconforming elements it was observed that the method converged for some mesh orientations but not for others (Bazeley et al. [1965]).

This sensitivity to mesh orientation led to the proposition of a numerical "patch test" by Irons (see Irons and Razzaque [1972]). The essential idea of their patch test was that in the limit as the mesh size approaches

zero the appropriate constant strain states should be representable on an arbitrary patch of elements adjacent to a given node. A formal mathematical statement of a patch test was subsequently developed by Strang [1973], and further refined by Stummel [1979]. At the same time, error estimates for nonconforming elements were being developed to further explain the performance of the method (Lascaux and LeSaint [1975]) and techniques for enforcing the smoothness requirements on the basis as constraints in the variational problem were being explored.

In the following sections we first summarize the convergence results and patch test and then examine the use of multiplier and particularly penalty methods for enforcing the smoothness constraints across the interfaces between elements.

2. CONVERGENCE

Let us consider a variational boundary-value problem of the form: find $u \in H$ such that

$$(2.1) \qquad a(u, v) = f(v) \qquad \forall v \in H$$

where $a(\cdot, \cdot)$ and $f(\cdot)$ denote the bilinear and linear functional for the variational statement.

The corresponding (nonconforming) approximate problem is: find $u_h \in H^h$, with $H^h \not\subset H$, such that

$$(2.2) \qquad a_h(u_h, v_h) = f(v_h) \qquad \forall v_h \in H^h$$

for

$$a_h(u_h, v_h) = \sum_{e=1}^{E} a_e(u_h^e, v_h^e)$$

where $a_e(\cdot, \cdot)$ is the functional on element Ω_e and E is the total number of elements in the discretization. Note that the effect of interface discontinuities between elements is ignored in the problem statement (2.2) and that this is the source of difficulties in nonconforming elements. An error analysis of the problem (2.2) reveals the following results (see, for instance, Ciarlet [1978],

$$(2.3) \qquad \|u - u_h\|_h \leq C[\inf_{v_h \in H^h} \|u - v_h\|_h + \sup_{w_h \in H^h} \frac{|f(w_h) - a_h(u, w_h)|}{\|w_h\|_h}]$$

where u is the exact solution, $\| \cdot \|_h$ is the norm associated with $a_h(\cdot, \cdot)$ in (2.2) and C is a constant independent of the mesh h. The first term on the right in (2.3) is the standard term in finite element error analysis and can be bounded by the interpolation estimate. The second term is an added contribution arising from the nonconformity of the subspace H^h. Selection of a "good" nonconforming element can then be rephrased as the problem of showing that the sup term in (2.3) is zero, or at least goes to zero with h, for the element in question. For example, in Wilson's nonconforming brick element for three-dimensional, second-order problems the sup term can be bounded by $C_1 h |u|_{2, \Omega}$.

3. PATCH TEST

The numerical patch test of Irons led to a mathematical formulation by Strang [1973] as follows: Given a variational boundary-value problem of order 2m, let P_m be a polynomial of degree m and let χ be a nonconforming basis function. Then to pass the patch test it is required that

$$(3.1) \qquad a(P_m, \chi) - a_h(P_m, \chi) = 0$$

For example, in the case of the Laplacian (Δu), m = 1 ,

$P_m \equiv P_1 = a + bx + cy$ and by Gauss theorem

$$(3.2) \qquad\qquad a(P_1, \chi) - a_h(P_1, \chi) = \oint \chi \frac{\partial P_1}{\partial n} ds$$

where the contour integral is over the exterior boundary and all interior
element boundaries. Hence, the test in (3.1) implies that the contour
integral in (3.2) vanishes.

We can sometimes show constructively that the boundary integral for each
individual element is zero — for example, integral contributions on opposite
sides of a particular nonconforming rectangle may cancel. It follows then
that the contour integral in (3.2) is zero, and we have a viable nonconform-
ing method. In other instances the contour integral on an element may not
vanish, but the accumulated contributions of the contour integrals to a patch
of several elements may vanish in which case the test (3.1) is again passed.
In fact, it is in this form that we may explicitly identify (3.1) as a "patch
test." Stummel [1979] introduced a "weak-closedness" argument to extend the
test to include certain pathological cases.

4. INTER-ELEMENT CONSTRAINTS

The origin of the difficulty is the lack of continuity of the basis
across the interfaces between elements. This point is emphasized by our
observations concerning the condition that the line integral in (3.2),
including all interelement boundaries, should vanish. We now consider
techniques for enforcing continuity across the interface between elements.

This continuity requirement can be interpreted as a constraint to be
enforced on each interface Γ_s (Figure 1). The most direct approach is to
imbed the continuity constraint in the existing basis, thereby modifying the
nonconforming basis to one that is conforming. For higher-order problems

this will lead to elements of high degree or composite elements, (Bell [1969], Clough and Tocher [1965]).
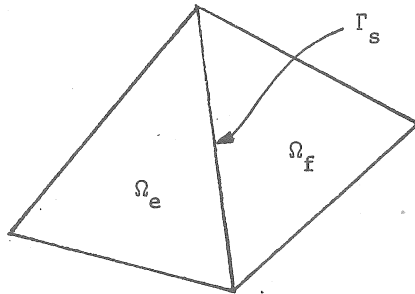

Figure 1. Adjacent elements Ω_e and Ω_f with typical constraints

$$[\![u]\!] = u_e - u_f = 0 \quad \text{or} \quad [\![\frac{\partial u}{\partial n}]\!] = \frac{\partial u_e}{\partial n} - \frac{\partial u_f}{\partial n} = 0 \quad \text{on} \quad \Gamma_s \; .$$

A second approach is to imbed the constraint in the variational statement by means of Lagrange multiplier techniques. The variational problem becomes that of finding the saddle point (u_h, λ_h) of the Lagrangian

$$(4.1) \qquad L(u_h, \lambda_h) = \frac{1}{2} \sum_{e=1}^{E} [a_e(u_h^e, u_h^e) - 2f(u_h^e)] + \sum_{s=1}^{S} \int_{\Gamma_s} \lambda_s [\![\gamma u]\!] ds$$

where $[\![\gamma(u)]\!]$ denotes the jump in the boundary trace term $\gamma(u)$ (de Veubeke [1968], Harvey and Kelsey [1971], Gallagher [1975]). The principal disadvantages of this technique are: (1) the multipliers enter as additional unknowns and the resulting algebraic system is correspondingly larger and less sparse; (2) the problem is a saddle-point problem on the spaces for approximating u and λ , and hence these spaces must be compatible in the sense of satisfying a discrete inf-sup condition if the method is to be stable and give convergent results. In fact, the method is equivalent to a hybrid method and the inf-sup stability condition implies that the degrees of the element bases for u_h and λ_h be consistent; (3) finally, the enforcement of interelement constraints using multipliers can introduce a subtle linear

dependence into the solution (Carey et al. [1982]) which may require the use

of special sparse solvers such as that in Argyris et al [1977].

Penalty methods provide still another means of imbedding constraints

in the variational statement. The essential ideas date back to the studies

by Courant [1962] who introduced a penalty term to enforce Dirchlet data on

the boundary in a potential problem. The problem statement for the

Dirchlet integral becomes

$$(4.2) \qquad \min_{v \in H^1} J = \frac{1}{2} \int_{\Omega} (\nabla v)^2 dx + \frac{1}{2\varepsilon} \int_{\partial\Omega} (v - g)^2 dS$$

where ε is the penalty parameter, and g is the Dirichlet data on the

boundary $\partial\Omega$ of domain Ω . As $\varepsilon \to 0$ the boundary penalty term is more

strongly enforced and the solution u_ε to (4.2) converges to the solution u

of the corresponding constrained variational problem. Note that no new

degrees of freedom are introduced in the associated finite element problem.

The method has been applied successfully for enforcing incompressibility

constraints in elasticity and viscous flow problems (see, for instance,

Hughes et al. [1976], Sani et al. [1981], and Carey and Krishnan [1982]).

Here we consider the use of penalty techniques for enforcing interele-

ment continuity of the basis. Let $J_h(v_h)$ denote the variational

functional for the nonconforming approximation,

$$(4.3) \qquad J_h(v_h) = a_h(v_h, v_h) - 2f(v_h) = \frac{1}{2} \sum_{e=1}^{E} [a_e(v_h^e, v_h^e) - 2f(v_h^e)]$$

Then the penalized problem has the form

$$(4.4) \qquad \min_{v_h \in H^h} J_\varepsilon(v_h) = \frac{1}{2} [\sum_{e=1}^{E} a_e(v_h^e, v_h^e) - 2f(v_h^e)] + \sum_{s=1}^{S} \frac{1}{2\varepsilon} \int_{\Gamma_s} [\![\gamma(v_h)]\!]^2 ds$$

where the penalized term enforces the condition that the jump $[\![\gamma(v_h)]\!]$ be

zero across interfaces Γ_s . For example, both the problems of deflection of

a clamped plate in bending and the stream function solution of steady viscous flow are fourth order and governed by the biharmonic equation. The penalized approximate problem using nonconforming elements with interface constraints on $\gamma(u_h) = \partial u_h/\partial n$ is

$$(4.5) \quad \min_{v_h \in H^h} \ J_\varepsilon(v_h) \ = \ \frac{1}{2} \sum_{e=1}^{E} \ [\int_{\Omega_e} \{(\Delta v_h^e)^2 - 2fv_h^e\}dx] + \sum_{s=1}^{S} \frac{1}{2\varepsilon} \int_{\Gamma_s} [\![\frac{\partial v_h}{\partial n}]\!]^2 \ ds$$

and taking variations, this implies the weak statement

$$(4.6) \quad \sum_{e=1}^{E} \int_{\Omega_e} \Delta u_h^e \ \Delta v_h^e dx + \sum_{s=1}^{S} \frac{1}{\varepsilon} \int_{\Gamma_s} [\![\frac{\partial u_h}{\partial n}]\!][\![\frac{\partial v_h}{\partial n}]\!]ds \ = \ \sum_{e=1}^{E} \int_{\Omega_e} fv_h^e dx$$

In the following numerical studies we shall see, however, that this approach fails to produce results that converge as the mesh size h is refined, and that the constraint condition is too strongly enforced in the discrete problem (4.6). Motivated by the success of approximate integration strategies for relaxing the penalty constraint. in other finite element applications, we introduce the reduced integration penalty problem for (4.6): find $u_h^\varepsilon \in H^h$ such that

$$(4.7) \quad \sum_{e=1}^{E} \int_{\Omega_e} \Delta u_h^e \ \Delta v_h^e dx + \sum_{s=1}^{S} \frac{1}{\varepsilon} I_s([\![\frac{\partial u_h}{\partial n}]\!][\![\frac{\partial v_h}{\partial n}]\!]) \ = \ \sum_{e=1}^{E} \int_{\Omega_e} fv_h^e dx$$

where $I_s(\cdot)$ denotes a Gauss quadrature formula for integrating the penalty term approximately. In matrix form, the penalty problem (4.7) reduces to finding the nodal solution $\underset{\sim}{\alpha}_\varepsilon$ to the linear algebraic system

$$(4.8) \qquad\qquad (\underset{\sim}{K} + \frac{1}{\varepsilon} \underset{\sim}{P})\underset{\sim}{\alpha}_\varepsilon \ = \ \underset{\sim}{F}$$

where $\underset{\sim}{\alpha}_\varepsilon$ are the degrees of freedom in the finite element expansion, $u_h(x) = \Sigma\alpha_j\phi_j(x)$, global basis $\{\phi_j\}$.

5. NUMERICAL EXAMPLE

In this illustrative numerical example we consider the above biharmonic problem for a square domain $\Omega = (0, 1) \times (0, 1)$ and a discretization of Hermite cubic triangles. The degrees-of-freedom for the element are u, u_x, u_y at each vertex and u at each centroid. The cubic is complete on each element but the global approximation is only C^0. The forcing function f is chosen so that we have the smooth solution

$$(5.1) \qquad u(x, y) = x^2 y^2 (1 - x^2)(1 - y^2)$$

The problem was solved on a sequence of uniformly refined meshes of right isosceles triangles with element side $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{16}$. The log of the error in the H^1 norm is plotted against log h for the exactly integrated penalty form (4.6) and an underintegrated form of (4.7) in Figure (2). The slope of the curve gives the rate of convergence. Since the error in the solution consists of a part due to the penalty approximation (dependent on ε) and a finite element approximation error (dependent on h) we take $\varepsilon = Ch^\sigma$, constant C with σ to be determined so that the two errors are of the same order and the accuracy is optimal. Following arguments similar to that of Babuška and Zlamal [1973] for (4.6) we show (Carey and Utku [1983]) that $\sigma = 3$ for (4.7) and the element in question. In the figure we plot results as a family of curves for different choices of C . We see that the fully-integrated method fails to converge for the meshes considered. Closer examination of the solution reveals that the approximate solutions approach $u = 0$. Note that in this problem the meshes considered extend to much finer resolution than one would use in engineering practice for such a problem.

When three-point (exact integration) and two-point (inexact) Gaussian quadrature are used for the interface constraint in (4.7) the same behavior

is seen. It is only when we further reduce the order of integration to a one-point rule that the convergent results in Figure 2 are obtained.

One can explain the success of this underintegrated penalty form by means of an equivalence theory relating the penalty method to a corresponding multiplier (hybrid) method. The discrete inf-sup condition for this equivalent hybrid method leads to a rank condition and hence a consistency condition on the approximation basis for the solution and multiplier where

$$(5.2) \qquad \lambda_h^\varepsilon(\xi_i) \; = \; \frac{1}{\varepsilon} \; [\![\frac{\partial u_h^\varepsilon}{\partial n} (\xi_i)]\!] \; , \quad \text{Gauss point} \;\; \xi_i$$

Let t be the degree of the equivalent multiplier so defined and $s = k - 1$ be the degree of $[\![\frac{\partial v_h}{\partial n}]\!]$. Then the method is stable if

$$(5.3) \qquad \begin{array}{l} t \; \leq s - 1 \; , \;\; s \;\; \text{odd} \\[2mm] t \; \leq s - 2 \; , \;\; s \;\; \text{even} \end{array}$$

In the present example $k = 3$ (cubic) so $s = 2$ and $t = 0$ for stability implies λ_h^ε is piecewise constant and the required integration rule is a one-point rule as observed experimentally above. Further theoretical details and numerical results are given in Carey, Kabaila, and Utku [1982] and Carey and Utku [1983].

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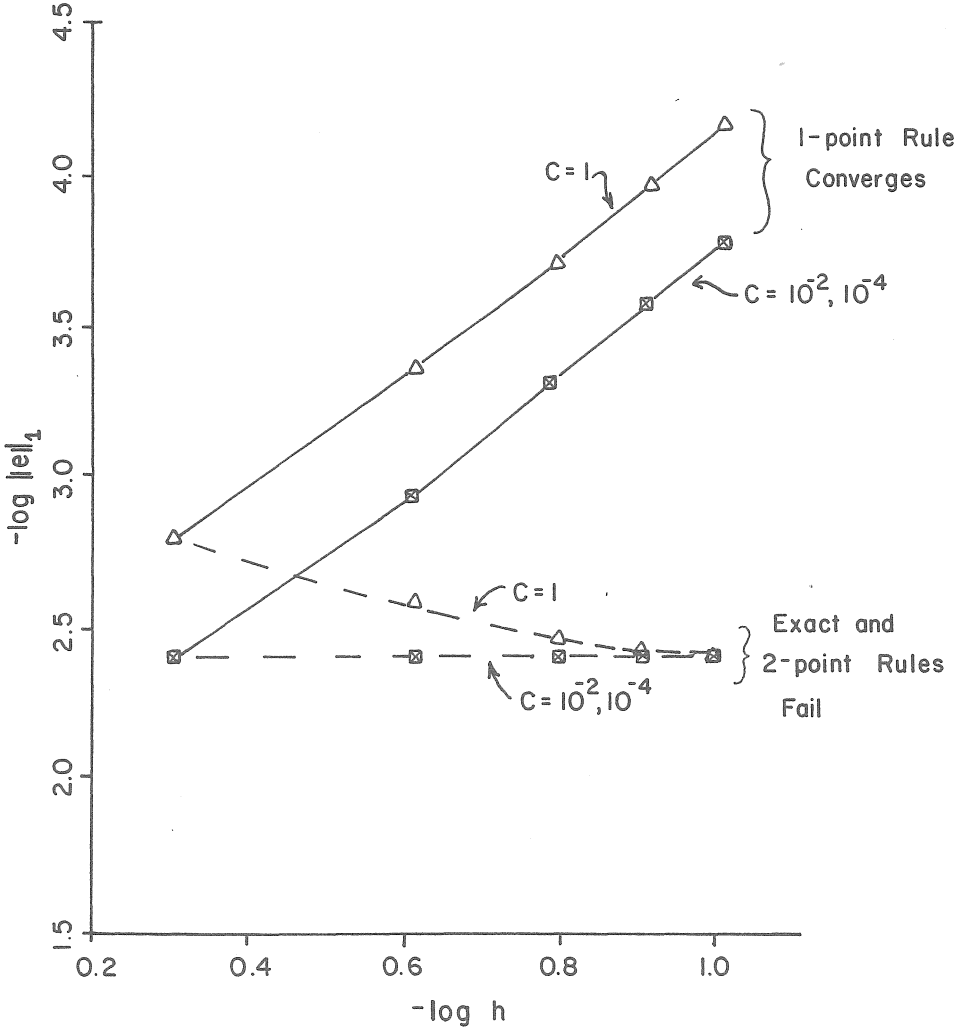
Figure 2. Performance of methods as the mesh is refined and for different choices of C in $\epsilon = Ch^3$.

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