THE DIRICHLET PROBLEM FOR THE MINIMAL SURFACE EQUATION

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0. INTRODUCTION

In this paper we consider the Dirichlet problem for the minimal surface equation. We assume that Ω is a bounded open set in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$ and that ϕ is a continuous function on $\partial\Omega$. Then we consider the problem :

(P) Find $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that

(i) $u = \phi$ on $\partial\Omega$,

(ii) u satisfies the minimal surface equation in Ω, that is,

$$\sum_{i=1}^{n} D_i \left(\frac{D_i u}{\sqrt{1+|Du|^2}} \right) = 0 \quad \text{in} \quad \Omega .$$

We shall consider two aspects of this problem : firstly, whether or not solutions exist and, secondly, the regularity of solutions.

1. EXISTENCE

A rather complete answer was obtained for the existence question by Jenkins and Serrin in 1968.

THEOREM 1 [JS]

(i) *If* $\partial\Omega$ *has nonnegative mean curvature everywhere then there is a solution to* (P) *for every* $\phi \in C^0(\partial\Omega)$.

(ii) If $\partial\Omega$ *has strictly negative mean curvature somewhere and* $\varepsilon > 0$ *then there exists* $\phi \in C^0(\partial\Omega)$ *with* $|\phi| \leq \varepsilon$ *such that there is no solution to* (P).

It should be noted that in the case $n = 2$, where the condition that $\partial\Omega$ has nonnegative mean curvature is equivalent to the convexity of Ω, the above results were known earlier (See [B], [F].)

We shall be concerned with the case where $\partial\Omega$ has negative mean curvature at some points. The above Theorem shows that we cannot expect to solve (P) for every continuous ϕ. However there are some functions ϕ for which there is a solution ; for example $\phi = 0$ has solution $u = 0$. Thus we may expect that if ϕ is sufficiently close to 0 in some norm then there should still be a solution to (P). This is indeed true provided we use an appropriate norm. We shall consider which norm is the most appropriate.

The first result in this direction was obtained by Korn in 1909.

THEOREM 2 [K]

If $\|\phi\|_{2,\alpha}$ *is sufficiently small there is a solution to* (P).

Korn's method of proof is to use the $C^{2,\alpha}$ estimates for solutions of Poisson's equation together with the contraction mapping principle. Using weak solutions and $C^{1,\alpha}$ estimates the same method shows that it is sufficient to assume $\|\phi\|_{1,\alpha}$ sufficiently small. It should be noted that the result of Jenkins and Serrin above shows that it is not sufficient just to have $\|\phi\|_0$ small

One of the consequences of the results to be given below is the following theorem which has also been proved independently by Chi-ping Lau in his thesis at the University of Minnesota.

THEOREM 3 [W1], [LA]

(i) *If* $\|\phi\|_{0,1}$ $(= \sup_{\partial\Omega} |\phi| +$ Lipschitz constant of $\phi)$ *is sufficiently small then* (P) *has a solution.*

(ii) *If* $0 < \alpha < 1$ *and* $\varepsilon > 0$ *then there is a function* ϕ *with* $\|\phi\|_{0,\alpha} < \varepsilon$ *such that* (P) *has no solution.*

Thus the $C^{0,1}$ norm is the correct one to work with. However it is in fact not crucial for existence that this norm be small. Indeed although $|\phi|$ must be small the Lipschitz constant of ϕ, $\mathrm{Lip}(\phi)$, may be quite large.

THEOREM 4 [W1]

(i) *Given* $K < \frac{1}{\sqrt{n-1}}$ *there is* $\varepsilon > 0$ *depending on* Ω, n *and* K *such that if* $|\phi| \leqq \varepsilon$ *and* $\mathrm{Lip}(\phi) \leqq K$ *then* (P) *has a solution.*

(ii) *Given* $K > \frac{1}{\sqrt{n-1}}$ *and* $\varepsilon > 0$ *there is a function* ϕ *with* $|\phi| \leqq \varepsilon$ *and* $\mathrm{Lip}(\phi) \leqq K$ *such that there is no solution of* (P).

The proof of (i) uses a barrier construction and the proof of (ii) involves a reduction to the consideration of positive solutions for Poisson's equation in conical domains. Strong use is made of an idea of Leon Simon [S] which involves the consideration of any solution as a graph over the boundary cylinder.

The results have been generalized, in joint work with Friedmar Schulz [SW], to include a wide class of elliptic equation (including those of minimal surface type) with the same sharp dependence on the Lipschitz constant again appearing.

2. REGULARITY

For the remainder of the paper we shall assume that $\partial\Omega$ has nonnegative mean curvature and so by the results of Jenkins and Serrin we know that (P) has a solution for any $\phi \in C^0(\partial\Omega)$. We shall consider the smoothness of the solution u. It is known that u is analytic in the interior of Ω and so we will be interested in the smoothness up to $\partial\Omega$. It is to be expected that the smoothness of u up to $\partial\Omega$ should increase as the smoothness of ϕ is increased and this is indeed the case. For example if $\phi \in C^{k,\alpha}(\partial\Omega)$ with $k \geq 2$ then the results of Jenkins and Serrin [JS] plus standard theory show that $u \in C^{k,\alpha}(\bar{\Omega})$ (see [GT]). More recently Lieberman [L1] and Giaquinta and Giusti [GG] have shown that if $\phi \in C^{1,\alpha}(\partial\Omega)$ then $u \in C^{1,\alpha}(\bar{\Omega})$. The remaining cases to be considered are when $\phi \in C^{0,\alpha}(\partial\Omega)$. Some results in this case have been given by Giusti in 1972 [G] (for more general equations see [L2]).

<u>THEOREM 5</u> [G]

(i) *If* $\partial\Omega$ *has strictly positive mean curvature and* $\phi \in C^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$ *then* $u \in C^{0,\alpha/2}(\bar{\Omega})$

(ii) *If* $\partial\Omega$ *has nonnegative mean curvature and* $\phi \in C^{0,1}(\partial\Omega)$ *then there exists* $\alpha > 0$ *such that* $u \in C^{0,\alpha}(\bar{\Omega})$.

Giusti also produced an example where $\partial\Omega$ has strictly positive mean curvature, $\phi \in C^{0,1}(\partial\Omega)$ but $u \notin C^{0,\alpha}(\bar{\Omega})$ for any $\alpha > \frac{1}{2}$. Thus (i) above is best possible.

In [W2] and [W3] these results are improved and generalized. For example it is shown that if the mean curvature of $\partial\Omega$ grows like $|x-x_0|^k$ near a point $x_0 \in \partial\Omega$ and $\phi \in C^{0,\alpha}(\partial\Omega)$ then u satisfies a Hölder condition at x_0 with exponent $\alpha/k+2$. This result is shown to be best possible and also corresponding results are shown using a general modulus of continuity rather than a Hölder condition.

At this stage it should be noted that a small change in the regularity of ϕ may result in a large change in the regularity of u. For example changing the regularity of ϕ from $C^{1,\varepsilon}$ to $C^{0,1}$ changes the known regularity of u from $C^{1,\varepsilon}$ to $C^{0,\frac{1}{2}}$ (at best). The next result shows that the regularity of u does depend continuously on the regularity of ϕ provided we take the Lipschitz constant of ϕ into account.

THEOREM 6 [W2]

(i) *Suppose* $\partial\Omega$ *has nonnegative mean curvature and* $0<\alpha<1$. *Then there exists a constant* $K(n,\alpha)$ *such that if* $\mathrm{Lip}(\phi)<K(n,\alpha)$ *then* $u \in C^{0,\alpha}(\bar{\Omega})$.

(ii) For any $K > K(n,\alpha)$ *there exists* ϕ *with* $\mathrm{Lip}(\phi) \leqq K$ *but such that* $u \notin C^{0,\alpha}(\bar{\Omega})$. (We assume here that $\alpha > \frac{1}{2}$ in the case of strictly positive mean curvature or $\alpha > \frac{1}{k+2}$ in the generalization mentioned above).

The proofs of these results are all by barrier constructions sometimes using the idea of Leon Simon mentioned above.

REMARKS

(i) The constants $K(n,\alpha)$ may be given explicitly. For example if $n = 2$, $K(2,\alpha) = \text{cotangent}\,(\frac{\pi\alpha}{2})$. It should be noted that they do not depend on Ω at all.

(ii) It is now possible to determine the best α for Giusti's Theorem 5 (ii). It will be the best of $\frac{1}{k+2}$ (k as above) and α such that $\mathrm{Lip}(\phi) < K(n,\alpha)$.

(iii) In all these results (including the existence questions) it is not really the Lipschitz constant of ϕ which is the crucial quantity but rather the size of any angles in the graph of ϕ. Thus if $\phi = \phi_1 + \phi_2$ where ϕ_1 is a C^1 function then we may use the Lipschitz constant of ϕ_2 instead of ϕ.

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