A SEMILINEAR ELLIPTIC BOUNDARY-VALUE PROBLEM DESCRIBING SMALL PATCHES OF VORTICITY IN AN OTHERWISE IRROTATIONAL FLOW

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SYNOPSIS

Let Ω be a bounded domain in $\mathbb{R}^2$. The study, begun in Keady[1981] and Keady and Kloeden[1984] of the boundary-value problem, for $(\lambda/k, \psi)$

$$-\Delta\psi \in \lambda H(\psi - k) \qquad \text{in} \qquad \Omega \subset \mathbb{R}^2 ,$$

$$\psi = 0 \qquad\qquad \text{on} \qquad \partial\Omega ,$$

is continued. Here Δ denotes the Laplacian, H is the Heaviside step function and one of λ or k is a given positive constant. The solutions considered always have $\psi > 0$ in Ω and $\lambda/k > 0$, and have cores

$$A = \{(x, y) \in \Omega \mid \psi(x, y) > k\} .$$

In the special case $\Omega = B(0, R)$, a disc, the explicit exact solutions are available. They satisfy

$$(*) \qquad\qquad (\psi_m - k)/k \to 0 \qquad \text{as} \qquad area(A) \to 0 ,$$

where ψ_m is the maximum of ψ over Ω. Here $(*)$ will be established for other domains.

An adaptation of the maximum principles of Gidas, Ni and Nirenberg [1979] is an important step in establishing the above result.

1. INTRODUCTION

The boundary-value problem studied in this paper arises in connection with the steady flow of an inviscid incompressible fluid with compact vortex cores.

For real numbers t define the set-valued Heaviside step-function H ,

$$H(t) = 0 \quad \text{if} \quad t < 0, H(0) = [0,1], H(t) = 1 \quad \text{if} \quad t > 0 \,.$$

Let Ω be a bounded planar domain with C^2 boundary and consider the boundary-value problem

$$\begin{aligned} -\Delta\psi \in \lambda H(\psi - k) \qquad & \text{in} \qquad \Omega\,, \\ \psi = 0 \qquad & \text{on} \qquad \partial\Omega\,, \end{aligned}$$

where λ , k are both positive. Problem (P) is to find, given one of λ or k , pairs $(\lambda/k, \psi)$ solving the preceding boundary-value problem.

The subset

$$A = \{(x,y) \in \Omega \mid \psi(x,y) > k\}\,,$$

is called the (vortex) core of such a solution.

To make the above precise requires the definition of 'solution'. We say $(\lambda/k, \psi)$ solves problem (P) if $\lambda/k > 0$ and ψ belongs to $\overset{\circ}{W}{}^2_1(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for $0 < \alpha < 1$ and if the boundary of A has zero measure.

Using slightly different techniques and with different restrictions on Ω ,A this problem was studied in Keady[1981] and Keady and Kloeden[1984], which are henceforth denoted as (I) and (II) respectively. The notation here will follow that of (I) and (II).

In (I) the asymptotic estimate (*) was established for doubly-symmetrised domains Ω. In (II) it was established for other domains but with restrictions on the cores A : as an example, A convex is a sufficient restriction.

The main results of (II) including the asymptotic estimate (*) are consequences of Theorem II.1.1. This shows that the points z_m belong to a compact subset $M(\Omega)$ of Ω constructed by means of domain folding arguments of Gidas, Ni and Nirenberg[1979]. This subset is defined in Section 3 of (II).

THEOREM II.1.1. *Let $(\lambda/k, \psi)$ be a solution with core A of problem (P) with ψ attaining its maximum value at z_m in A. Then*

$$z_m \in M(\Omega) ,$$

where $M(\Omega)$ is a compact subset of Ω , and hence

$$distance(z_m \ , \ \partial\Omega) \geq d > 0 ,$$

where

$$d = inf\{| z - z^* | \ | \ z \in \partial\Omega \ , z^* \in M(\Omega)\} .$$

For symmetrised domains $M(\Omega)$ lies on the axis of symmetry of Ω.

For strictly convex domains Ω Theorem II.1.1. is true with $M_0(\Omega)$ defined in Section 2 below, replacing $M(\Omega)$.

In this paper it will be shown that it is possible to use more information from Gidas, Ni and Nirenberg[1979] than that given in Theorem II.1.1. The technique is similar to that in de Figueiredo, Lions and Nussbaum[1981]. The information will be used to establish the asymptotic estimate (*) when Ω is strictly convex with a uniform positive lower bound on the curvature on the boundary of Ω. As this restriction on Ω is known, by the results of (I) and (II), not to be necessary, the proofs of various intermediate steps are arranged, where possible, to be independent of it.

2. RESULTS OF GIDAS, NI AND NIRENBERG

The subset $M(\Omega)$ containing the points z_m of Theorem II.1.1 is defined by means of domain folding arguments of Gidas, Ni and Nirenberg[1979]. Let D denote an arbitrary bounded domain in $\mathbb{R}^2$ with smooth boundary and define the subsets

$$D_+(t) = \{(x,y) \in D \mid y > t\} ,$$

$$D_-(t) = \{(x,y) \in D \mid y < t\} ,$$

and

$$D_+^r(t) = \{(x, 2t - y) \mid (x, y) \in D_+(t)\} ,$$

for any real number t . Also define the real number $t_*(D)$ as in (II). For convex D the definition is,

$$t_*(D) = sup\{t \in \mathbb{R} \mid D_-(t) \subseteq D_+^r(t)\} .$$

The following lemma is established in II:

THEOREM II.3.3. *Let $(\lambda/k, \psi)$ be a solution of problem (P). Then for all (x,y) in Ω with $y < t_*(\Omega)$,*

$$\psi(x, 2t - y) - \psi(x, y) > 0 ,$$

$$\psi_y(x, y) > 0 .$$

These results also apply for domain foldings relative to a general direction $(cos\theta, sin\theta)$, in which case the subset

$$D_+(t, \theta) = \{(x, y) \in D \mid x cos\theta + y sin\theta > t\} ,$$

replaces the subset $D_+(t)$ corresponding to the angle $\theta = \pi/2$. Similar generalizations give the subsets $D_-(t, \theta)$ and $D_+^r(t, \theta)$ and the number $t_*(D, \theta)$. A subset $M_0(D)$ of D is then defined as

$$M_0(D) = \bigcap_{0 \leq \theta \leq 2\pi} \overline{D_+(t_*(D, \theta), \theta)} .$$

From Theorem II.3.3 (or Theorem II.3.4) it follows that

$$z_m \in M_0(\Omega) .$$

When Ω is a symmetrised domain $M_0(\Omega)$ is a subset of the axis of symmetry of Ω.

When Ω is a convex domain $M_0(\Omega)$ is convex. When Ω is strictly convex with the boundary of Ω of class C^2 and with the curvature bounded away from zero, we will say that Ω belongs to U. For Ω a member of U

$$d(M_0(\Omega), \partial\Omega) > 0 .$$

De Figueiredo, Lions and Nussbaum[1982] observe (in their Theorem 1.1) the following:

THEOREM 2.1. *Let Ω belong to U. There exist $\pi/2 > \theta > 0$ and $\tau_0 > 0$ with θ and τ_0 depending only on Ω with the following property.*

Let z_0 be a point on the boundary of Ω and $n = n(z_0)$ be the unit outward normal from Ω at z_0. Then for any solution of problem (P) , $\psi(z_0 - \tau\nu)$ is nondecreasing for τ increasing between 0 and τ_0, for unit vectors ν satisfying $\langle \nu, n(z_0)\rangle \geq cos\theta$. With a suitable choice of θ, τ_0, this is true with the same θ, τ_0 for any z_0 on the boundary of Ω.

COROLLARY 2.1. *There exist positive numbers ε and θ, with $\theta < \pi/2$, depending only on Ω, such that the following, called property (F-L-N), holds:*

For all z in Ω with $d(z, \partial\Omega) < \varepsilon$ there exists a cone (or wedge) $I(z)$ with vertex at z, with semi-vertex angle θ and with axis $\{z - sn(z) \mid s \geq 0\}$ such that

$$\psi(\xi) \geq \psi(z) \quad \forall \xi \in I(z) \text{ with } d(\xi, \partial\Omega) < \varepsilon \ .$$

Here n(z) is the negative of the direction of the shortest line from z to the boundary of Ω.

3. THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS

Several ideas for the proof of Theorem 3.1 below were suggested by Norm Dancer.

THEOREM 3.1. *Let $(\lambda_\nu/k_\nu, \psi_\nu)$ be a sequence of solutions of problem (P) with cores A_ν such that $area(A_\nu)$ tends to zero. Suppose that*

$$limsup\ (\psi_{\nu m}/k_\nu) = c < \infty \qquad \text{as} \qquad area(A_\nu) \to 0 \ ,$$

(where $c \geq 1$ is independent of ν and depends only on Ω) . Then $c = 1$, that is

$$(\psi_{\nu m} - k_\nu)/k_\nu \to 0 \qquad \text{as} \qquad area(A_\nu) \to 0 \ .$$

Proof. If the electrostatic capacity, defined in (I) and (II), $cap(A,\Omega)$ tends to zero as $area(A)$ tends to zero then we already have, as in (I), $(\psi_m - k)/k$ tends to zero. Hence assume, taking subsequences if necessary that $cap(A,\Omega)$ tends to $c_* > 0$. Hence $area(A)$ tends to zero implies λ/k tends to infinity.

Consider a subsequence with $z_{\nu m}$ tending to z_*. We now choose origins, different for different ν, at $z_{\nu m}$. Rescale the coordinates so that

$$X = \sqrt{\lambda}x\ , \qquad Y = \sqrt{\lambda}y\ ,$$

and write

$$\psi_\lambda(X,Y) = \psi(x,y)/k \quad \text{and} \quad \Omega_\lambda = \{(\sqrt{\lambda}x, \sqrt{\lambda}y) \mid (x,y) \in \Omega\}\ .$$

We now have

$$-\Delta\psi_\lambda = -\frac{\partial^2\psi_\lambda}{\partial X^2} - \frac{\partial^2\psi_\lambda}{\partial Y^2} = H(\psi_\lambda - 1) \quad \text{a.e. in} \quad \Omega_\lambda\ .$$

Next consider any compact subset of $\mathbb{R}^2$. We assert that (a subsequence of) ψ_λ with λ tending to infinity converges uniformly on that compact subset. We will denote the limit function by v. It will suffice to establish the result on any closed disc B, with radius R.

The proof of the assertion of the previous paragraph is, by Arzela-Ascoli, to use the uniform boundedness that $\psi_\lambda \le c$ and uniform equicontinuity from

$$|\nabla\psi_\lambda| \le C_1(c + (R+1)^2)\ .$$

This last statement follows from Theorem 3.9 and Problem 8.4 of Gilbarg and Trudinger, where their Ω is taken as B_1, a ball of radius $(R+1)$ with the same centre as B. Then, with $d_z = dist(z, B_1)$

$$d_z\,|\nabla\psi_\lambda| \le C_1(sup\,|\psi_\lambda| + sup\ d_z^2)$$
$$\le C_1(c + (R+1)^2)\ .$$

Thus, since for z belonging to B , $1 \le d_z$,

$$|\nabla\psi_\lambda| \le C_1(c + (R+1)^2)\ .$$

Next we observe that we can find a subsequence such that ψ_λ tends to v uniformly on compact subsets of $\mathbb{R}^2$. Let $B_{m,n}$ be the unit disk centred at (m,n) with integer m, n. Then index the plane lattice of pairs of integers so that $B_j = B_{m,n}$. We start with a sequence $\psi_{1,\lambda}$ tending to v on B_1. Having found a sequence $\psi_{j,\lambda}$ tending to v on the union of B_k with $1 \leq k \leq j$ we refine it so that $\psi_{j+1,\lambda}$ tends to v on the union of B_k with $1 \leq k \leq (j+1)$. We are left with a sequence ψ_λ tending to v uniformly on every B_j , which suffices.

Each of the ψ_λ is superharmonic. Thus v is also superharmonic on $\mathbb{R}^2$. Also $v \leq c$ and hence v is constant on $\mathbb{R}^2$. Since $\psi_\lambda(0) > 1$, $v(0) \geq 1$.

Since

$$-\Delta\psi_\lambda = H(\psi_\lambda - 1)$$

we expect

$$-\Delta v \in H(v-1) \qquad \text{in} \qquad \mathbb{R}^2 \ .$$

We have (Gilbarg and Trudinger p67),

$$\psi_\lambda(z) = \frac{1}{\mid B \mid}\int_{B(z,R)} \psi_\lambda + \frac{1}{2\pi}\int_{B(z,r)} \Theta H(\psi_\lambda - 1) \ ,$$

where

$$\Theta(r,R) = log(R/r) - 1/2(1-(r/R)^2) \ .$$

Assume z is such that $v(z) < 1$ at some point z, and hence in some $B(z,R)$. Then for λ sufficiently large the $\psi_\lambda < 1$ in $B(z,R)$. Thus v is harmonic at z.

Assume z is such that $v(z) > 1$. Then, similarly,

$$-(\Delta v)(z) = 1 \ .$$

This suffices for our purposes.

We know that v is constant with $v(0) \geq 1$. If $v(0) > 1$ then $-\Delta v = 1$ in $\mathbb{R}^2$ and hence v is not constant, a contradicition. Thus $v(0) = 1$.

The conclusion is that

$$((\psi_m - k)/k) = \psi_\lambda(0) - 1 \rightarrow v(0) - 1 = 0$$

as asserted.

Remark. For the uniform convergence of ψ_λ to v we used an interior gradient estimate. Is there a global gradient estimate of the form

$$max \mid \nabla\psi \mid^2 \leq C\lambda(\lambda area(A))^p \quad \text{in} \quad \Omega ,$$

$$or \quad max \mid \nabla\psi \mid^2 \leq C\lambda\psi_m \quad \text{in} \quad \Omega \ ?$$

In a convex domain Ω Sperb[1981] has shown that

$$\mid \nabla\psi \mid^2 \leq \mid \nabla\psi \mid^2 + 2\lambda(\psi - k)_+ \leq 2\lambda(\psi_m - k) \leq 2\lambda\psi_m$$

so that the second form is true with $C = 2$.

THEOREM 3.2. *Let $(\lambda_\nu/k_\nu, \psi_\nu)$ be a sequence of solutions of problem (P) . Suppose that*

$$liminf \ d(A_\nu, \partial\Omega) = c_A > 0 \quad \text{as} \quad \nu \to \infty$$

(where c_A is independent of ν and depends only on Ω) . Then

$$limsup \ (\psi_{\nu m}/k_\nu) = c < \infty \quad \text{as} \quad \nu \to \infty ,$$

(where $c \geq 1$ is obviously independent of ν).

Proof. Let $\varepsilon < c_A$ and

$$N_\varepsilon(\Omega) = \{z \in \Omega \mid d(z, \partial\Omega) \geq c_A - \varepsilon\} .$$

For ν sufficiently large A_ν is a subset of $N_\varepsilon(\Omega)$. Thus

$$cap(A_\nu, \Omega) \leq cap(N_\varepsilon(\Omega), \Omega) < \infty$$

and

$$limsup \ cap(A_\nu, \Omega) \leq cap(N_0(\Omega), \Omega) < \infty .$$

Using inequalities from the torsion problem for $\Psi = \psi - k$ in (components of) A we have

$$\psi_m - k \leq \frac{1}{4\pi}\lambda area(A).$$

Since $\lambda area(A) = k\ cap(A, \Omega)$ the result of the previous paragraph gives

$$limsup\ (\psi_{\nu m}/k) \leq 1 + cap(N_0(\Omega), \Omega)/(4\pi)\ .$$

THEOREM 3.3. *Let $(\lambda_\nu/k_\nu, \psi_\nu)$ be a sequence of solutions of problem (P) with cores A_ν such that $area(A_\nu)$ tends to zero.*

Suppose that there exist positive numbers ε and θ , with $\theta < \pi/2$, depending only on Ω such that property (F-L-N) holds:

Then

$$liminf\ d(A_\nu, \partial\Omega) = c_A \geq \varepsilon \quad \text{as} \quad area(A_\nu) \to 0\ .$$

Proof. Suppose there exist points of A_ν within ε of the boundary of Ω (If not, for all ν, there is nothing to prove.) Let z_ν be such that

$$d(z_\nu, \partial\Omega) = min\{d(z, \partial\Omega) \mid z \in A_\nu\}\ ,$$

and let

$$L_\nu = \varepsilon - d(z_\nu, \partial\Omega)\ .$$

Then

$$L_\nu^2 \tan\theta \leq area(A_\nu)$$

and hence L_ν tends to zero as $area(A_\nu)$ tends to zero, and thus the distance between z_ν and the boundary of Ω tends to ε. This establishes the theorem.

Remark. We expect that the distance between z_ν and $M_0(\Omega)$ tends to zero but have not yet attempted to prove this stronger statement.

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