

RADON SINGULAR INTEGRALS

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This report will describe some recent work done jointly with D.H. Phong [4], [5]. Only a sketch of the main ideas, the background, and the motivation will be presented here. The reader who is interested in further details may consult the cited literature.

1. DEFINITIONS

We assume we are dealing with a (compact) smooth $n+1$ dimensional Riemannian manifold M . Suppose that for each point P in M we are given a co-dimension one smooth sub-manifold, M_P , so that $P \in M_P$. We suppose also that for each P we are given a Calderón-Zygmund kernel $K(P, \cdot)$ on M_P , with pole at P . We shall assume that the mappings $P \rightarrow M_P$, and $P \rightarrow K(P, \cdot)$ are smooth.

With these we form the "Radon singular integral", mapping (smooth) functions on M to functions on M , given by

$$(Tf)(P) = \langle K(P, \cdot), f \Big|_{M_P} \rangle = \int_{M_P} K(P, Q) f(Q) d\sigma_P(Q),$$

i.e. with the restriction of f to M_P integrated against the Calderón-Zygmund kernel $K(P, \cdot)$, using the measure $d\sigma_P$ which is induced from the Riemannian volume element on M .

Together with the singular integral T we also consider a corresponding maximal function. To define it let $B(P, \delta)$ denote the intersection of M_P with the geodesic ball centred at P of radius δ . Then set

$$M(F)(P) = \sup_{0 < \delta \leq 1} \frac{1}{|B(P, \delta)|} \int_{B(P, \delta)} |f(Q)| d\sigma_P(Q).$$

We shall deal with the following problem:

Question: Are T and M bounded on L^p , $1 < p < \infty$?

In retrospect one can see that this problem has in reality a long history as we shall explain momentarily (however the appellation Radon singular integrals, suggested by the relation with Radon transforms, seems to be new). The initial motivation for studying this question came from non-isotropic singular integrals; the early results then lead to a host of further questions in particular in real-variable theory. A recent impetus has been the $\bar{\partial}$ -Neumann problem.

2. FIRST EXAMPLES

We shall give several examples of Radon singular integrals and their associated maximal functions.

(1) Here $M = \mathbb{R}^{n+1}$ (although M is not compact!), and for each $P = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$, take $M_P = P + \mathbb{R}^n$, where $\mathbb{R}^n = (x_1, x_2, \dots, x_n, 0)$, i.e. M_P is the affine hyperplane passing through P which is perpendicular to $(0, 0, \dots, 0, 1)$. In this case both T and M are easily reducible to the standard singular integrals and maximal functions on \mathbb{R}^n .

(2) This example is already more representative of the situation we consider. Here we take M to be \mathbb{R}^2 , and define

$$(Tf)(x) = \text{P.V.} \int_{-\infty}^{\infty} f(x - \gamma(t)) \frac{dt}{t}$$

where $\gamma(t)$ is the parabola $\gamma(t) = (t, t^2)$. (Here $M_0 = (t, t^2)$, and $M_P = P + M_0$.) This operator is translation-invariant, and so its boundedness on $L^2(\mathbb{R}^2)$ can be proved by showing that

$m(\xi) = \text{P.V.} \int_{-\infty}^{\infty} e^{i\gamma(t) \cdot \xi} \frac{dt}{t}$ is a bounded function, as was done by

Fabes in 1965 [1].

(3) The theory of Hilbert transforms (and associated maximal functions) for curves was then greatly extended thanks to the work of Nagel, Rivière, and Wainger; (see [6] for a systematic account of the subject and its history). What came to light from this work was the important role of curvature in these problems.

3. ROTATIONAL CURVATURE

The theory above used the curvature of the manifolds M_P ; but these sub-manifolds were merely translations of a fixed M_0 . On the other hand it will be important to consider situations where the M_P are "flat", but as they vary with P they "rotate". This notion is subsumed in the following idea of "rotational curvature". Suppose $\Phi(P, Q) = 0$, is a defining function of the manifolds $\{(P, Q) \mid Q \in M_P\}$. Let us assume that $d_P \Phi(P, Q) \neq 0$ and $d_Q \Phi(P, Q) \neq 0$. We form the $(n+2) \times (n+2)$ "Monge-Ampère determinant"

$$\det \begin{pmatrix} \Phi(P, Q) & d_P \Phi \\ d_Q \Phi & d_{PQ}^2 \Phi \end{pmatrix},$$

and require that it be non-vanishing for $P = Q$.

This condition is closely related to a condition of Guillemin and Sternberg [3], used in their generalization of the invertibility of the Radon transform. A less intrinsic, but more workable version is as follows. One can find appropriate coordinate systems so that (locally) $M = \mathbb{R}^{n+1} = \{(t, x)\}$ with $t \in \mathbb{R}$, $x \in \mathbb{R}^n$; and if we set $P = (t, x)$, then $M_P = \{(s, y), s = t + S(t, x, y)\}$, where the function S satisfies $S(t, x, x) = 0$, and

$$\det \left(\frac{\partial^2 S(t, x, y)}{\partial x_i \partial y_k} \right) \neq 0 \quad \text{for } x = y .$$

Note: It is easy to see that example 2) above satisfies this curvature condition, but of course example 1) does not.

4. FURTHER EXAMPLES

(4) Let $H^m = \mathbb{C}^m \times \mathbb{R} = \{(z, t)\}$ be the Heisenberg group, with multiplication formula

$$(z, t) \cdot (\omega, s) = (z + \omega, t + s + 2\text{Im}z \cdot \bar{\omega}) .$$

Consider the distribution K given by $K = L(z) \delta_t$ where δ_t is the Dirac delta function in the t -variable, and L is a standard Calderón-Zygmund kernel (homogenous of degree $-2m$) on \mathbb{C}^m . The operator T is given by $Tf = f * K$, with convolution on the Heisenberg group. Here $n = 2m$, M_0 is the hyperplane $\{(z, 0)\}$, and M_P , $P = (\omega, s)$, is the right-translate (Heisenberg group multiplication!) of M_0 by (ω, s) .

In this setting one can exploit "twisted convolution", as was done by Geller and the author [2]; the success of this method suggests that the use of the Fourier transform in only one variable might be a useful tool in the non-translation invariant case, an idea we shall adopt below.

(5) This example is in reality a generalization of the previous one and is of basic importance. Let \mathcal{D} be a smooth bounded domain in \mathbb{C}^{m+1} , and let r be a defining function for \mathcal{D} . One then constructs a function $\psi(z, \omega)$ which is (almost) analytic in z , (almost) anti-analytic in ω , and so that $\psi(z, z) = r(z)$. We set $\partial\mathcal{D} = M$, and if $P = \omega \in b\mathcal{D}$, we take $M_P = \{z \in \partial\mathcal{D} \mid \text{Im} \psi(z, \omega) = 0\}$. Then the condition of non-vanishing curvature described in §3 above is

equivalent with the non-degeneracy of the Levi-form associated to \mathcal{D} . It also turns out that the Radon singular integrals in this setting are crucial operators if one wants to obtain sharp estimates for the $\bar{\partial}$ -Neumann problem.

5. THE THEOREM

Theorem. Suppose $n \geq 2$ and $\{M_p\}$ has non-zero curvature as described in §4. Then both T and M are bounded on L^p , $1 < p < \infty$.

The idea of the proof (of the L^2 boundedness of T) is to write T as a pseudo-differential operator in one variable (hiding the other variables by considering functions which take their values in appropriate Hilbert spaces). More specifically, one can write

$$(Tf)(t) = \int_{-\infty}^{\infty} a(t, \lambda) e^{i\lambda t} \hat{f}(\lambda) d\lambda$$

where

$$\hat{f}(\lambda) = \hat{f}(\lambda, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} f(t, x) dt$$

(that is $\hat{f}(\lambda)$ takes its values in $L^2_x(\mathbb{R}^n)$). Similarly $(Tf)(t)$ takes its values in that space. Also $a(t, \lambda)$ takes its values in the space of operators from $L^2(\mathbb{R}^n)$ to itself. The definition of $a(t, \lambda)$ is

$$a(t, \lambda)(f)(x) = \int_{\mathbb{R}^n} K(t, x; x-y) e^{i\lambda S(t, x, y)} f(y) dy.$$

$K(t, x; \cdot)$ is a Calderón-Zygmund kernel, depending smoothly on (t, x) and having compact support in (t, x) . The main problem is then of showing that the oscillations of $e^{i\lambda S}$, as $\lambda \rightarrow \infty$, and those of K do not interfere, and in fact work together for the good. One can show that

$$(*) \quad \left\| \left(\frac{\partial}{\partial t} \right)^k \left(\frac{\partial}{\partial \lambda} \right)^\ell a(t, \lambda) \right\|_{\text{op.}} \leq A(1 + |\lambda|)^{k/2 - \ell/2}, \quad 0 \leq k + \ell < n$$

and then apply a version of the Calderón-Vaillancourt theorem. To prove (*) the condition

$$\det \left(\frac{\partial^2 S(t, x, y)}{\partial x_i \partial y_k} \right) \neq 0, \quad x = y,$$

plays a decisive role.

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