ON A MODIFIED MEAN CURVATURE FLOW

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In recent years various evolution equations have been studied in differential geometry. In a landmark paper Hamilton [4] studied the deformation of Riemannian metrics on compact manifolds in direction of their Ricci-curvature:

$$\frac{d}{dt} g_{ij} = -2Ric_{ij} + \frac{2}{n} r g_{ij} \tag{1}$$

Here g_{ij} is the Riemannian metric, Ric_{ij} is the Ricci curvature and $r = \fint R d\mu$ is the average of the scalar curvature on the manifold. This is a weakly parabolic system and Hamilton showed that on a three dimensional manifold any initial metric of positive Ricci-curvature flows into a metric of constant positive curvature when evolved by equation (1). This result was later extended to higher dimensions in [5] and [7].

In [6] we studied the mean curvature flow, that is an evolution equation for hypersurfaces M^n embedded in $\mathbb{R}^{n+1}$: Let the initial hypersurface M_0^n be given locally by some diffeomorphism

$$F_0 : U \subset \mathbb{R}^n \longrightarrow F_0(U) \subset M_0 \subset \mathbb{R}^{n+1} ,$$

then we want to find a whole family $F(.,t)$ of diffeomorphisms corresponding to hypersurfaces M_t such that the evolution equation

$$\begin{aligned} &\frac{d}{dt} F(\vec{x},t) = -H(\vec{x},t) \, .\nu(\vec{x},t) \qquad \vec{x} \in U \\ (2)\qquad & \\ &F(.,0) = F_0 \end{aligned}$$

is satisfied. Here H is the mean curvature on M_t and ν is the unit normal to M_t, such that the hypersurfaces are flowing in direction of the mean curvature vector. Again (2) is a weakly parabolic system and the resulting evolution equation for the second fundamental form on M_t has a structure similar to the evolution equation for the curvature tensor resulting from (1). Whereas in (1) one has to assume that the intrinsic curvature of the initial metric is sufficiently positive in order to obtain a convergence result, for the mean curvature flow in (2) one has to assume that the initial hypersurface is uniformly convex. We have from [6]:

THEOREM 1 *Let $n \geq 2$ and assume that M_0 is uniformly convex, i.e. the eigenvalues of the second fundamental form are strictly positive everywhere. Then the evolution equation (2) has a smooth solution M_t on a finite time interval $0 \leq t < T$ and the hypersurfaces M_t contract to a single point 0 as $t \to T$. The shape of M_t becomes more and more spherical as $t \to T$, i.e. homothetic expansions of M_t with a fixed area around 0 converge to a round sphere.*

Corresponding results for curves in $\mathbb{R}^2$ were shown in [2] and [3], and in [8] and [1] a similar result was established for hypersurfaces moving along their Gauss-curvature.

Here we study an evolution equation which keeps the volume enclosed by the hypersurfaces M_t constant without having to rescale:

$$\frac{d}{dt} F(\vec{x},t) = (h(t) - H(\vec{x},t)) \,.\, \nu(\vec{x},t)$$

(3)

$$F(.,0) = F_0$$

where $h = \fint H d\mu$ is the mean value of the mean curvature on M_t. The enclosed volume is clearly constant since $\int h - H d\mu = 0$ and we will see that the total area of M_t is decreasing: one can expect M_t to converge to a solution of the isoparametric problem. We show that this is indeed the case if the initial hypersurface is uniformly convex:

THEOREM 2 *Let* $n \geq 2$ *and assume that* M_0 *is uniformly convex. Then the evolution equation* (3) *has a smooth solution* M_t *for all times* $0 \leq t < \infty$ *and* M_t *converges smoothly to a round sphere as* $t \to \infty$.

The strategy of the proof aims at obtaining a uniform upper bound for the mean curvature on M_t . We show that the curvature can only blow up in a uniform way, then contradicting the constancy of the enclosed volume. The necessary estimates are more involved than in the standard mean curvature flow since we don't have an *a priori* lower bound for the mean curvature and since h introduces a global quantity into all evolution equations. We sketch the proof of Theorem 2, the details will appear elsewhere.

A Evolution equations and convexity properties

Let $g = \{g_{ij}\}$ and $A = \{h_{ij}\}$ denote the metric and the second fundamental form on M_t respectively. Using the Gauss-Weingarten relations one can then deduce from (3) evolution equations for g and A:

$$\frac{d}{dt} g_{ij} = 2(h-H)h_{ij}$$

$$\frac{d}{dt} h_{ij} = \Delta h_{ij} - 2Hh_{im}h^m_j + hh_{im}h^m_j + |A|^2 h_{ij} ,$$

where $|A|^2 = g^{ij}g^{k\ell}h_{ik}h_{j\ell}$ is the norm of the second fundamental form squared and Δ is the (time dependent) Laplace operator on M_t . The evolution equation for g implies that the area of the hypersurfaces is decreasing, we have

$$\frac{d}{dt}|M_t| = -\int_{M_t}(H-h)^2 d\mu \quad . \tag{4}$$

Also, the evolution equation for h_{ij} together with a maximum principle for parabolic systems developed by Hamilton in [4] implies that the inequality $h_{ij} > 0$ is preserved, in other words:
A uniformly convex initial hypersurface stays uniformly convex for all times. We can even show

LEMMA 1 *If for some* $0 < \epsilon \leq \frac{1}{n}$ *we have on* M_0 *the inequality*

$$h_{ij} \geq \epsilon Hg_{ij} \quad , \quad H > 0$$

then this inequality is preserved with the same ϵ *for all times* t *where the solution* M_t *of* (3) *exists.*
Note however that we do not yet have a lower bound for the mean curvature.

B A pinching estimate

In this step we show that the eigenvalues of the second fundamental form come close together at least at those points where the mean curvature is large. Let $\kappa_1,\ldots,\kappa_n$ denote the eigenvalues of A and consider the quantity

$$|A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i<j}(\kappa_i-\kappa_j)^2$$

which measures how much the κ_i's differ from each other. We prove

LEMMA 2 *There is* $C_0 < \infty$ *and* $\delta > 0$ *depending only on* M_0 *and* ϵ *such that*

$$|A|^2 - \frac{1}{n}H^2 \leq C_0 H^{2-\delta}$$

holds on M_t *for all times* t .

Proof: We want to bound the function

$$f_\sigma = \frac{|A|^2 - H^2}{H^{2-\sigma}}$$

for small enough $\sigma > 0$. The evolution equation for h_{ij} implies evolution equations for $|A|^2$ and H and after some lengthy calculations we derive

$$\frac{d}{dt} f_\sigma \leq \Delta f_\sigma + \frac{2(1-\sigma)}{H} \langle \nabla_\ell H, \nabla_\ell f_\sigma \rangle - \frac{\epsilon}{H^{2-\sigma}} |\nabla H|^2$$

(5)

$$+ \sigma |A|^2 f_\sigma - 2\epsilon^2 hHf_\sigma$$

where we also used the convexity property from Lemma 1. If we now had an inequality like $h \geq \delta H$, the Lemma would immediately follow for small enough σ since $|A|^2 \leq H^2$. Unfortunately there seems to be no easy way to obtain such an estimate and we have to use the negative $|\nabla H|^2$ term on the right hand side to absorb the positive term $\sigma |A|^2 f_\sigma$ somehow. The main tool is a Poincaré - type inequality for convex hypersurfaces proven in [6]:

LEMMA 3 *Let* $p \geq 2$. *Then for any* $\eta > 0$ *and any* $0 \leq \sigma \leq \frac{1}{2}$ *we have*

$$n\epsilon^2 \int f_\sigma^p H^2 d\mu \leq (2\eta p + 5) \int \frac{1}{H^{2-\sigma}} f_\sigma^{p-1} |\nabla H|^2 d\mu$$

$$+ \eta^{-1}(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu \quad .$$

Using this inequality we first obtain from (5) bounds for high L^p norms of f_σ where $p \sim \sigma^{-2}$. The sup-norm bound is then obtained from the Sobolev inequality and an iteration method.

C Derivative estimates

Here we show that the gradient of the mean curvature becomes small at points where H is large.

Lemma 4 For any $\eta > 0$ there is a constant C_η depending only on M_0 and η but not on T such that for all $0 \leq t \leq T$ we have

$$|\nabla H|^2 \leq \eta \max_{t\in[0,T]} \max_{M_t} H^4 + C_\eta .$$

Proof: We compute an evolution equation for $|\nabla H|^2$ and obtain

$$\frac{d}{dt} |\nabla H|^2 \leq \Delta |\nabla H|^2 + 8H(H+h)|\nabla A|^2 . \tag{6}$$

The idea is then to add enough of the evolution equation of

$$g = H(H+h)(|A|^2 - \frac{1}{n} H^2)$$

to this inequality in order to absorb the bad term on the RHS of (6) and to control the bad terms in the evolution equation of g with Lemma 3. We omit the details.

Lemma 4 can now be used to compare the mean curvature at different points of the surfaces M_t and we can show that the maximum of H can only tend to infinity if the minimum of H blows up also. But then all principal curvatures on M_t would go to infinity by Lemma 1 contradicting the constancy of the enclosed volume.

It follows that the mean curvature is uniformly bounded for all times and it is then easy to obtain estimates for all higher derivatives of the second fundamental form as well. Thus the solution of (3) exists for all times $0 \leq t < \infty$ and one has only to show that M_t converges to a round sphere as $t \to \infty$.
To see this, note that (4) implies

$$\int_0^\infty \int (H-h)^2 d\mu dt \leq |M_0|$$

and our uniform estimates then show that

$$\sup_{M_t} |H-h| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \quad .$$

Now we obtain from (5) that

$$|A|^2 - \frac{1}{n} H^2 = \frac{1}{n} \sum_{i<j} (\kappa_i - \kappa_j)^2$$

decays exponentially as $t \longrightarrow \infty$ and the result follows.

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