REGULARITY THEOREMS FOR ELLIPTIC EQUATIONS WITH NON-SMOOTH COEFFICIENTS

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0. PRELIMINARIES

We are concerned with the elliptic equation

$$(1) \qquad Lu(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha u(x),$$

where the a_α's are not infinitely differentiable but merely are locally in some Besov space $B^s_{p,q}$ or Triebel space $F^s_{p,q}$. Hereafter we assume that all functions and distributions are defined on $\mathbb{R}^n$. As

$$Lu(x) = \tau(x,D)u(x) = (2\pi)^{-n} \int \tau(x,\xi)e^{ix\cdot\xi}\hat{u}(\xi)d\xi, \quad u \in S$$

where

$$(2) \qquad \tau(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)(i\xi)^\alpha,$$

and

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi}u(x)dx$$

is the Fourier transform of u, one is led to study pseudo-differential operators (ψdos) whose symbols $\sigma(x,\xi)$ (not necessarily of the form (2)) are not smooth in x. In fact, motivated by applications to equation (1), we proved in [Bu1] the following result.

THEOREM 0 (cf. [Bul, Theorem 3]). *Assume that* $\rho > \rho_B$ *(resp.* ρ_F*) and*

$$\|\partial^\beta \sigma(\cdot,\xi)\|_{B^\rho_{\infty,\infty}} \leqq C_\beta (1 + |\xi|)^{-|\beta|}$$

for any multi-index β *and for all* $\xi \in \mathbb{R}^n$. *Then* $\sigma(x,D)$ *is bounded on* $B^s_{p,q}$ *(resp.* $F^s_{p,q}$*).*

Unfortunately, there is no good symbolic calculus for the type of symbols in Theorem 0. In fact, Bony [Bo] noted that it is impossible to include in the same algebra all the differential operators of constant coefficients as well as the operators of multiplying by functions in $B^\rho_{\infty,\infty}$. To overcome this difficulty, Bony [Bo] replaced the ordinary multiplication by an operation he called para-multiplication, a version of which, as given by Meyer [M], will be presented next.

Let ψ be a function in $\mathcal{S}$ such that supp $\hat{\psi} = \{1/2 \leqq |\xi| \leqq 2\}$, and $\Sigma_{j=-\infty}^{\infty} \hat{\psi}(2^{-j}\xi) = 1$ for any $\xi \neq 0$. Let ψ_j, $j = 0,1,2,\ldots$, be such that

$$\hat{\psi}_j(\xi) = \hat{\psi}(2^{-j}\xi), \; j = 1,2,\ldots,$$

and

$$\hat{\psi}_0(\xi) + \sum_{j=1}^{\infty} \hat{\psi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n.$$

Let a be a bounded function. Then, the para-multiplication by a is defined by

$$\pi(a,u) = \sum_{0\leqq j\leqq k-3} (\psi_j * a)(\psi_k * u)$$

$$= \sum_{k=3}^{\infty} \left(\sum_{j=0}^{k-3} \psi_j * a \right) (\psi_k * u), \; u \in \mathcal{S}.$$

It is easily seen that $\pi(a,\cdot)$ is a ψdo whose symbol σ_a is given by

$$\sigma_a(x,\xi) = \sum_{k=3}^{\infty} m_k(x)\hat{\psi}(2^{-k}\xi),$$

where

$$m_k = \left(\sum_{j=0}^{k-3} \psi_j\right)*a.$$

As $\Sigma_{j=0}^{k-3} \hat{\psi}_j(\xi) = \hat{\psi}_0(2^{-(k-3)}\xi)$, we see that

$$\|m_k\|_\infty \leq C\|a\|_\infty.$$

Here we adopt the convention that C is a constant which may be different from one occurrence to the next one, and which may depend on the particular parameters appearing in the context. Noting that $\text{supp}\ \hat{m}_k \subset \{|\xi| \leq 2^k\}$, we derive from the above inequality and Bernstein's theorem (cf. [P, Chap. 3, Lemma 1]) that

$$\|\partial^\alpha m_k\|_\infty \leq C2^{k|\alpha|} .$$

Thus, σ_a satisfies

$$(3) \qquad |\partial_x^\alpha \partial_\xi^\beta \sigma_a(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{|\alpha|-|\beta|}, \ x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n,$$

i.e., $\sigma_a \in S^0_{1,1}$. It is well-known that ψdos whose symbols are in $S^0_{1,1}$ are not even bounded on L^2. The boundedness property of these operators is investigated in the next section.

Next, we define the spaces necessary for our study. Following Peetre [P] and Triebel [T] (cf. also [Bu2]), we define

$$B^s_{p,q} = \{f \in \mathcal{S}' \,\Big|\, \|f\|_{B^s_{p,q}} = \Big(\sum_{j=0}^{\infty} (2^{js}\|\psi_j * f\|_p)^q\Big)^{1/q} < \infty\},$$

$$F^s_{p,q} = \{f \in \mathcal{S}' \,\Big|\, \|f\|_{F^s_{p,q}} = \Big\| \Big(\sum_{j=0}^{\infty} (2^{js}|\psi_j * f(\cdot)|)^q\Big)^{1/q}\Big\|_p < \infty\},$$

where $-\infty < s < \infty$, $0 < p$, $q \leqq \infty$, and $p < \infty$ for F-space; s, p and q will be as above unless otherwise indicated.

The above two scales of function spaces contain many function spaces appearing in the literature, e.g., the generalized Sobolev space, denoted by L^p_s by Meyer [M] $(1 < p < \infty)$, coincides with $F^s_{p,2}$; the Hölder-Zygmund space $C^\rho = B^\rho_{\infty,\infty}$ $(\rho > 0)$; the local Hardy space $h^p = F^0_{p,2}$.

1. OPERATORS WITH SYMBOLS IN $S^0_{1,1}$

Our aim is to prove the following theorem.

THEOREM 1 *Assume that* $\sigma \in S^0_{1,1}$, *i.e.,* σ *satisfies (3).*

(i) If $s > \max(0, n(1-1/p))$, *then* $\sigma(x,D)$ *is bounded on* $B^s_{p,q}$.

(ii) If either

$$0 < p \leqq 1, \quad p < q \leqq \infty \quad \textit{and} \quad s > n(1/p - p/q),$$

or

$$1 < p < \infty, \quad p < q \leqq \infty \quad \textit{and} \quad s > n(1/p - 1/q),$$

then $\sigma(x,D)$ *is bounded on* $F^s_{p,q}$.

Proof We begin with the proof of (i). We follow the method used to prove Theorem 0. Assume first that σ is an elementary symbol, i.e.,

$$\sigma(x,\xi) = \sum_{j=0}^{\infty} m_j(x)\hat{\phi}_j(\xi),$$

$$\text{supp } \hat{\phi}_0 \subset \{|\xi| \leqq 2^N\}, \quad \phi_0 \in S,$$

$$\text{supp } \hat{\phi}_j \subset \{2^{j-N} \leqq |\xi| \leqq 2^{j+N}\}, \ \phi_j \in S, \ j = 1,2,\ldots ,$$

for some positive integer N, and

(4) $$\|\partial^\alpha m_j\|_\infty \leqq C_\alpha 2^{j|\alpha|}, \ j = 0,1,2,\ldots .$$

Then

$$\sigma(x,D)f = \sum_{j=0}^{\infty} m_j(\phi_j * f), \ f \in S.$$

and for each $k = 0,1,2,\ldots,$

(5) $$\psi_k * \sigma(x,D)f = \sum_{j,\ell} \psi_k * [(\psi_\ell * m_j)(\phi_j * f)]$$

$$= \sum_{j,\ell} \psi_k * f_{j,\ell}.$$

By considering the supports of $\hat{\psi}_k$ and $\hat{f}_{j,\ell}$, we derive that there exists a positive integer m such that $\psi_k * f_{j,\ell} = 0$ except for those j and ℓ satisfying

(6) $$0 \leqq j \leqq (k - mN)_+ = k^* \text{ and } k^* \leqq \ell \leqq k + mN,$$

(7) $$0 \leqq \ell \leqq k^* \text{ and } k^* \leqq j \leqq k + mN,$$

or

(8) $$j,\ell \geqq k^* \text{ and } |j-\ell| \leqq mN + 2.$$

Denote the corresponding sum in the right-hand side of (5) by S_k^1, S_k^2 and S_k^3, respectively. We shall give the estimates for $k \neq 0$, as the case $k = 0$ can be similarly handled. For each ℓ as in (6),

$$S^1_{k,\ell}(x) = \sum_{j=0}^{k^*} |\psi_k * f_{j,\ell}(x)|$$

$$\leq \sum_{j=0}^{k^*} \int |\psi(y)||\phi_j * f(x-2^{-k}y)||\psi_\ell * m_j(x-2^{-k}y)|dy$$

$$\leq C(\psi,\lambda)\left\{ \sum_{j=0}^{k^*} \|\psi_\ell * m_j\|_\infty \phi^*_{j\lambda} f(x)\right\}, \ \lambda > n/p,$$

where $\phi^*_{j\lambda} f$ is defined as in [Bu2, p.587]. Now, it is easily seen that

$$\|\psi_\ell * m_j\|_\infty \leq C2^{-2\ell h}\|\Delta^h m_j\|_\infty$$

$$\leq C_h 2^{2(j-\ell)h} \qquad \text{(by (4))},$$

and thus,

$$\text{(9)} \qquad S^1_{k,\ell}(x) \leq C \sum_{j=0}^{k^*} 2^{2(j-\ell)h} \phi^*_{j\lambda} f(x).$$

If $0 < p < 1$, then, choosing $2h > s$, we derive from the above inequality (9) that

$$2^{ks}\|S^1_{k,\ell}\|_p \leq C\{\sup_{\substack{0\leq j\leq k^* \\ k^*\leq \ell\leq k+mN}} 2^{2(j-k+k-\ell)h+(k-j)s}\} \times$$

$$\times \left\{\sum_{j=0}^{\infty} (2^{js}\|\phi^*_{j\lambda} f\|_p)^p\right\}^{1/p}$$

$$\leq C\|f\|_{B^s_{p,p}}$$

by a maximal inequality (cf. [Bu2, Theorem 2.2]). As a similar estimate holds for k = 0, we obtain

$$(10) \qquad \sup_k 2^{ks} \|S_k^1\|_p \leq C\|f\|_{B^s_{p,p}}, \quad 0 < p < 1.$$

On the other hand, if $1 \leq p \leq \infty$, then with h as above, we see that

$$2^{ks}\|S^1_{k,\ell}\|_p \leq C \sum_{j=0}^{k*} 2^{2(j-\ell)h+(k-j)s} 2^{js} \|\phi^*_{j\lambda} f\|_p$$

$$\leq C\|f\|_{B^s_{p,1}},$$

and thus,

$$(11) \qquad \sup_k 2^{ks}\|S^1_k\| \leq C\|f\|_{B^s_{p,1}}, \quad 1 \leq p \leq \infty.$$

Next, we turn to the estimate for S^2_k. For each j as in (7),

$$(12) \qquad \left|\sum_{\ell=0}^{k*} \psi_k * f_{j,\ell}(x)\right|$$

$$= \left|\int \psi(y)\left[\left(\sum_{\ell=0}^{k*} \psi_\ell\right) * m_j(x-2^{-k}y)\right](\phi_j * f)(x-2^{-k}y)\,dy\right|$$

$$\leq C\left\|\sum_{j=0}^{k*} \psi_\ell\right\|_1 \|m_j\|_\infty \phi^*_{j\lambda} f(x)$$

$$\leq C\phi^*_{j\lambda} f(x)$$

by (4) and the fact that $\Sigma_{j=0}^{k*} \hat{\psi}_\ell(\xi) = \hat{\psi}_0(2^{-k*}\xi)$. Thus, it is obvious that S^2_k satisfies inequalities similar to (10) and (11).

Finally, as,

$$(13) \qquad S_k^3 = \sum_{j=k^*}^{\infty} \psi_k * \left[\left(\left(\sum_{|j-\ell| \leqq mN+2} \psi_\ell \right) * m_j \right) (\phi_j * f) \right],$$

an argument similar to that used in the estimate for S_k^2 shows that

$$\|S_k^3\|_p \leqq C \sum_{j=k^*}^{\infty} \|\phi_j * f\|_p, \ 1 \leqq p \leqq \infty,$$

and thus,

$$(13)' \qquad \sup_k 2^{ks} \|S_k^3\|_p \leqq C(\sup_{j \geqq k^*} 2^{(k-j)s}) \|f\|_{B^s_{p,1}}$$

$$\leqq C \|f\|_{B^s_{p,1}}, \ 1 \leqq p \leqq \infty,$$

as $s > 0$ and $j \geqq k^* = (k - mN)_+$. To estimate S_k^3 in the case $0 < p < 1$, note that

$$\operatorname{supp} \hat{\psi}_k \subset \{|\xi| \leqq 2^{j+mN+1}\},$$

$$\operatorname{supp}[\ldots]^{\wedge} \subset \{|\xi| \leqq 2^{j+mN+4}\}.$$

Hence, it follows from a convolution lemma [P, Chap.11, Lemma 8] that

$$\|\psi_k * [\ldots]\|_p^p \leqq C 2^{jn(1-p)} \|\psi_k\|_p^p \|(\Sigma_\ell \ \psi_\ell) * m_j\|_p^p \|\phi_j * f\|_p^p$$

$$\leqq C 2^{jn(1-p)+kn(p-1)} \|\phi_j * f\|_p^p \ .$$

Thus,

$$2^{ks} \|S_k^3\|_p \leqq C \left(\sum_{j=k^*}^{\infty} 2^{(k-j)(sp-n(1-p))} 2^{jsp} \|\phi_j * f\|_p^p \right)^{1/p}$$

$$\leqq C \|f\|_{B^s_{p,p}}$$

as $sp - n(1-p) > 0$ and $k - j \leqq mN$. Consequently, S_k^3 also satisfies inequalities similar to (10) and (11).

Now, combining the above estimates for S_k^1, S_k^2 and S_k^3, we derive that $\sigma(x,D)$ is bounded from $B^s_{p,p}$ into $B^s_{p,\infty}$ if $0 < p < 1$ and $s > n(1/p - 1)$, and $\sigma(x,D)$ is bounded from $B^s_{p,1}$ into $B^s_{p,\infty}$ if $1 \leqq p \leqq \infty$ and $s > 0$. Hence, $\sigma(x,D)$ is bounded on $B^s_{p,q}$ if $s > \max(0, n(1/p - 1))$ by real interpolation (cf. [Bu2, Theorem 3.3(i)]). The proof of (i) for elementary symbols is thus complete.

We now turn to the proof of (ii) in the case when σ is elementary. First, observe that (9) implies

$$\sup_{k\geqq 0} 2^{ks}|S_k^1(x)| \leqq C \sup_{j\geqq 0} 2^{js}\phi^*_{j\lambda} f(x). \tag{14}$$

Next, by (10) we see that S_k^2 also satisfies an inequality similar to (14). Finally, it follows from (13) that

$$\begin{aligned}
2^{ks}|S_k^3(x)| &= 2^{ks}\Big|\sum_{j=k^*}^{\infty} \int \psi(y)[(\Sigma_\ell\ \psi_\ell)*m_j(x - 2^{-k}y)] \times \\
&\qquad\qquad \times (\phi_j*f)(x - 2^{-k}y)dy\Big| \\
&\leqq 2^{ks} \sum_{j=k^*}^{\infty} \|(\Sigma_\ell\ \psi_\ell)*m_j\|_\infty \int |\psi(y)|(1 + 2^{j-k}|y|)^\lambda \frac{|\phi_j*f(x-2^{-k}y)|}{(1+2^j|2^{-k}y|)^\lambda}\, dy \\
&\leqq C\left(\sum_{j=k^*}^{\infty} 2^{ks}2^{(j-k)\lambda}\phi^*_{j\lambda} f(x)\right) \\
&\leqq C\left(\sum_{j=k^*}^{\infty} 2^{(j-k)(\lambda-s)}\right)\left(\sup_{j\geqq 0} 2^{js}\ \phi^*_{j\lambda} f(x)\right) \\
&\leqq C \sup_{j\geqq 0} 2^{js}\phi^*_{j\lambda} f(x) \quad \text{for} \quad \lambda > s > n/p\ .
\end{aligned}$$

Hence, it follows from a maximal inequality (cf. [Bu2, Theorem 2.2]) that $\sigma(x,D)$ is bounded on $F^{s}_{p,\infty}$ if $s > n/p$. Now, if p, q and s satisfy the assumptions of (ii) of the theorem, then we can choose s_0, s_1, p_0, p_1, θ such that

$$s_0 < s < s_1,\ p_0 < p < p_1,\ 0 < \theta < 1,$$

$$s_0 > \max(0, n(1-1/p_0)),\ s_1 > n/p_1,$$

$$s = (1-\theta)s_0 + \theta s_1,\ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\quad \frac{1}{q} = \frac{1-\theta}{p_0}.$$

Part (i) of the theorem implies that $\sigma(x,D)$ is bounded on $F^{s_0}_{p_0,p_0}$ $(= B^{s_0}_{p_0,p_0})$, while the above implies that $\sigma(x,D)$ is bounded on $F^{s_1}_{p_1,\infty}$. Thus, the desired result (ii) follows from complex interpolation ([T, Theorem 2.4.7 (ii)]).

The conclusion of the theorem for general symbols follows from that for elementary symbols by a standard method (cf. [C-F] or [Bu1, Proof of Theorem 3]).

2 REGULARITY THEOREMS FOR DIFFERENTIAL EQUATIONS

We return to equation (1). Assume that Lu = f, and that

$$\text{(15)}\qquad \begin{cases} \text{the } a_\alpha\text{'s, } u \text{ and } f \text{ are locally} \\ \text{in } B^{s}_{p,q} \text{ (resp. } F^{s}_{p,q}) \text{ at } x_0, \text{ where} \\ s > m + n/p \text{ (resp. } s > m + n/p,\ 0 < p < q \leqq \infty). \end{cases}$$

As our aim is to show that u is locally in $B^{s+m}_{p,q}$ (resp. $F^{s+m}_{p,q}$) at x_0, by multiplying by appropriate C^{∞}_{0}-functions, we may assume that the a_α's, u and f are in $B^{s}_{p,q}$ (resp. $F^{s}_{p,q}$). We shall give details only for the B-space case, because the other case can be similarly handled. By the Sobolev embedding theorem,

(16) $$u_\alpha = \partial^\alpha u \in B^{s-|\alpha|}_{p,q} \subset B^{s-m}_{p,q} \subset B^{s-m-n/p}_{\infty,\infty} = B^r_{\infty,\infty},$$

$$(r = s - m - n/p > 0),$$

so that Lu is defined pointwise. Now,

(17) $$Lu = \sum_\alpha a_\alpha \partial^\alpha u = \sum_\alpha \pi(a_\alpha, u_\alpha) + \sum_\alpha \pi(u_\alpha, a_\alpha) + \sum_\alpha R(a_\alpha, u_\alpha).$$

As each $u_\alpha \in L^\infty$ by (16), $\pi(u_\alpha, \cdot)$ is a ψdo with symbol in $S^0_{1,1}$, and thus,

(18) $$\sum_\alpha \pi(u_\alpha, a_\alpha) \in B^s_{p,q}$$

by Theorem 1. On the other hand, for each α,

$$R(a_\alpha, u_\alpha) = \sum_{|j-k|<3} (\psi_j * a_\alpha)(\psi_k * u_\alpha)$$

$$= \sum_{k=0}^\infty \left(\sum_{j=(k-2)_+}^{k+2} \psi_j * a_\alpha \right) (\psi_k * u_\alpha),$$

and hence, $R(a_\alpha, \cdot)$ is a ψdo whose symbol is given by

$$\sigma_\alpha(x,\xi) = \sum_{k=0}^\infty m_{k,\alpha}(x) \hat{\psi}_k(\xi),$$

where

$$m_{k,\alpha} = \sum_{j=(k-2)_+}^{k+2} (\psi_j * a_\alpha).$$

As $a_\alpha \in B^s_{p,q} \subset B^{m+r}_{\infty,\infty}$ (cf. (16)), we derive from Bernstein's theorem that

$$\|\partial^\gamma m_{k,\alpha}\|_\infty \leqq C2^{k(|\gamma|-m-r)}.$$

Consequently,

$$|\partial_x^\gamma \partial_\xi^\beta \sigma_\alpha(x,\xi)| \leqq C_{\gamma,\beta}(1 + |\xi|)^{-(m+r)-|\beta|+|\gamma|},$$

i.e., $\sigma_\alpha \in S_{1,1}^{-m-r}$. Therefore, it follows from Theorem 1 that

$$(19) \qquad R(a_\alpha,u_\alpha) \in B_{p,q}^{s-|\alpha|+m+r} \subset B_{p,q}^{s+r} .$$

Combining (17), (18) and (19), we obtain

$$\sum_\alpha \pi(a_\alpha,u_\alpha) = f - \sum_\alpha \pi(u_\alpha,a_\alpha) - \sum_\alpha R(a_\alpha, u_\alpha)$$

$$= g \in B_{p,q}^s.$$

Letting $v = (I - \Delta)^{m/2}u \in B_{p,q}^{s-m}$, we derive the following pseudo-differential equation

$$Av = \sum_\alpha \pi(a_\alpha,\cdot)\circ\partial^\alpha\circ(I-\Delta)^{-m/2}v = g.$$

Assume now that (x_0,ξ_0) is non-characteristic with respect to L, i.e.,

$$(20) \qquad \sum_{|\alpha|=m} a_\alpha(x_0)(i\xi_0)^\alpha \neq 0.$$

Then, as the symbol of A is given by

$$\sigma_A(x,\xi) = \sum_{|\alpha|\leqq m}\sum_{k=3}^{\infty}\left[\left(\sum_{j=0}^{k-3}\psi_j\right)*a_\alpha(x)\right]\times(i\xi)^\alpha(1+|\xi|^2)^{-m/2}\hat{\psi}(2^{-k}\xi),$$

it follows from (20) that

$$\text{(21)} \qquad \liminf_{\lambda\to\infty} |\sigma_A(x_0,\lambda\xi_0)| > 0.$$

Also, it is easy to verify that

$$\|\partial^\beta \sigma_A(\cdot,\xi)\|_{B^{m+r}_{\infty,\infty}} \leqq C_\beta(1+|\xi|)^{-|\beta|},$$

and for each fixed ξ,

$$\operatorname{supp} \hat{\sigma}_A(\eta,\xi) \subset \{|\eta| \leqq \tfrac{1}{2}|\xi|\}.$$

Thus, σ_A is in the class B_{r+m} defined by Meyer. This fact, (21) and [M, Proposition 4] imply that there exist $\tau \in S^0_{1,1}$, $\rho \in S^{-m-r}_{1,1}$, $\theta \in C_0^\infty$, $\mu \in C^\infty$ such that

$$\theta(x_0) = 1,\ \mu(\xi_0) \neq 0$$

$$\mu(\lambda\xi) = \mu(\xi) \text{ for } |\xi| \geqq R_0 \text{ and } \lambda \geqq 1,$$

and

$$\tau(x,D)\circ A = \theta(x)\mu(D) + \rho(x,D).$$

As $\tau(x,D)Av \in B^s_{p,q}$ and $\rho(x,D)v \in B^{s+r}_{p,q}$ by Theorem 1, it follows that $\theta(x)\mu(D)v \in B^s_{p,q}$, i.e., v is micro-locally in $B^s_{p,q}$ at (x_0,ξ_0). Further, if L is elliptic at x_0, then we can repeat the above argument for every direction ξ_0, and conclude that v is locally in $B^s_{p,q}$ at x_0, which implies that u is locally in $B^{s+m}_{p,q}$ at x_0. Thus, we have proved the following theorem.

THEOREM 2 *Assume that* L *is elliptic at* x_0, $Lu = f$, *and the assumptions (15) at the beginning of §2 are satisfied. Then the solution* u *is locally in* $B^{s+m}_{p,q}$ *(resp.* $F^{s+m}_{p,q}$*) at* x_0.

3 REMARKS AND FURTHER RESULTS

REMARK 1 Some cases of Theorem 2 have been known. In [B-R, Theorem 2.2], the result is proved for the space $B^s_{2,2}$ ($= F^s_{2,2} = H^s$) by the use of a different class of symbols. On the other hand, Theorem 2 for $B^s_{2,2}$, $B^s_{\infty,\infty}$ and $F^s_{p,2}$ ($1 < p < \infty$) are implicit in the works of Bony [Bo] and Meyer [M]. As seen from the proof of Theorem 2, the main tool, besides the symbolic calculus developed by Meyer, is Theorem 1, and in [M] Meyer showed that ψdos with symbols in $S^0_{1,1}$ are bounded on $F^s_{p,2}$ ($= L^p_s$ in his notation), $1 < p < \infty$, $s > 0$, and thus Theorem 2 is valid for $F^s_{p,2}$, $1 < p < \infty$, $s > n/p$, without the restriction $p \leqq 2$. Meyer's proof of the boundedness of ψdos relied on an inequality due to Paley (randomization) [M, Lemma 4], and it seems not possible to extend his arguments to the case $q \neq 2$. By complex interpolation of his result and ours (Theorem 1), one can remove the restriction $p \leqq q$ in Theorem 1 in some cases, and hence, Theorem 2 is true on any resulting space obtained by such interpolation.

REMARK 2 It is also a routine matter to extend the result of Bony and Meyer (cf. [M, Théorème 6]) on non-linear equations to our space $B^s_{p,q}$ and $F^s_{p,q}$, because the key tool is again Theorem 1. In fact, part of our result, Theorem 1(i) and the application to non-linear equations, has been also given in the author's talk [Bu3].

REMARK 3 This remark concerns with the extension of the results to weighted spaces. Theorem 0 has been extended to weighted spaces, where the weight function is in the class A_∞ of Muckenhoupt (cf. [Bu 2, Remark 3.4(c)]). The proof of Theorem 1 has been done in a way that it can be extended to some weighted spaces. In fact, the estimates for S_k^1 and S_k^2 are based on maximal inequalities and hence are readily extended to weighted spaces via the results in [Bu2]. As for S_k^3, if $1 < p < \infty$ and $w \in A_p$, then (13) implies that

$$\|S_k^3\|_{p,w} \leqq \sum_{j=k^*}^{\infty} \|M([\ldots])\|_{p,w}$$

$$\leqq C \sum_{j=k^*}^{\infty} \|[\ldots]\|_{p,w}$$

by the weighted estimate for the Hardy maximal function. (Here M denotes the Hardy maximal function.) Thus, (13)' holds also for the weighted case and hence, it follows that the weighted version of Theorem 1(i) is valid if $1 < p < \infty$, $0 < q \leqq \infty$ and $w \in A_p$ (by the interpolation theorem in [Bu2]). Consequently, we see that Theorem 2 is true for $B_{p,q}^{s,w}$ if furthermore $w \in \mathcal{M}_d$ and $s > m + d/p$ (the last two assumptions are made to ensure that we still have Sobolev embedding theorem (cf. [Bu2, Theorem 2.6 (v)]), so that $Lu = f$ is defined pointwise). It remains an open question to extend Theorems 1 and 2 to other cases (e.g., $w \in A_\infty$, $0 < p \leqq 1$, etc.).

REFERENCES

[B-R] M. Beals and M. Reed, *Microlocal regularity theorems for nonsommoth pseudodifferential operators and applications to nonlinear problems*, Trans. Amer. Math. Soc. 285 (1984), 159-184.

[Bo] J-M. Bony, *Cacul symbolique et propagation des singularités pour les équations aux dérivées partielles non lineaires*, Ann. Scient. Éc. Norm. Sup. 14 (1981), 209-246.

[Bu1] H-Q. Bui, *On Besov, Hardy and Triebel spaces for* $0 < p \leqq 1$, Ark. Mat. 21 (1983), 169-184.

[Bu2] H-Q. Bui, *Weighted Besov and Triebel spaces: Interpolation by the real method*, Hiroshima Math. J. 12 (1982), 581-605.

[Bu3] H-Q. Bui, *Some recent topics in function spaces*, Lecture given at the meeting on "Potential theory and related topics", Karuizawa, Japan, 1982.

[C-M] R.R. Coifman and Y. Meyer, *Au-delà des operateurs pseudo-differentiels*, Astérisque *57*, Soc. Math. France, 1978.

[M] Y. Meyer, *Remarques sur un théorème de J.M. Bony*, Prépublications, Univ. Paris-Sud, Orsay, France, 1980.

[P] J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Press, Durham, 1976.

[T] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel. Boston. Stuttgart, 1983.

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