SINGULAR INTEGRALS AND MAXIMAL FUNCTIONS ON CERTAIN LIE GROUPS

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INTRODUCTION

Let T be a convolution operator on $L^p(\mathcal{R}^n)$, $1 \leq p < \infty$ whose operator norm is denoted $||T||_p$. Then two basic results in the calculus of convolution operators are these.

PROPOSITION. *(i) If $1 \leq p < \infty$ and T is bounded on $L^p(\mathcal{R}^n)$, then T is also bounded on $L^{p'}(\mathcal{R}^n)$, and*

$$||T||_p = ||T||_{p'}.$$

(ii) If T is bounded on $L^p(\mathcal{R}^n)$, then T is bounded also on $L^2(\mathcal{R}^n)$, and

$$||T||_2 \leq ||T||_p.$$

In fact, (ii) is a consequence of (i) because of interpolation.

Now if we pass to a general locally compact Hausdorff group G with left invariant Haar measure, we may consider a kernel K and the corresponding operator $T_K = K\star$:

$$T_K f(x) = K \star f(x) = \int_G K(y) f(y^{-1}x)\, dy.$$

It was proved some time ago that if $1 \leq p \leq 2$, and T_K is bounded on $L^p(G)$, and G is *amenable*, then T_K is bounded on $L^2(G)$, and

$$||T||_2 \leq ||T||_p.$$

This is due to C. S. Herz [1].

Amenable groups share certain properties with abelian locally compact groups, viz. the possibility of constructing certain kinds of bounded local units. They include the compact, the nilpotent and the solvable Lie groups. However, noncompact semi-simple Lie groups such as $SL^2(\mathcal{R})$ are *not* amenable. This is borne out dramatically by the following result of N. Lohoué [2].

THEOREM. *Let G be a noncompact semi-simple Lie group with finite center. Suppose $1 < p_0 < \infty$. Then there is a positive measure μ on G which convolves L^{p_0} into L^{p_0} but does not map $L^p(G)$into $L^p(G)$ for any other p.*

Problem of asymmetry If G is a locally compact group, $1 \leq p < \infty$, and K is a kernel on G such that T_K maps $L^p(G)$into $L^p(G)$, does K also convolve $L^{p'}(G)$into $L^{p'}(G)$? If it does, do we have

$$||T_K||_p = ||T_K||_{p'}.$$

Note that if G is a finite group, then the answer to the first question is in the affirmative. However, D. Oberlin [3] showed a number of years ago that if G is the dihedral group of order 8, then there exist kernels which have *different* convolution norms on the spaces $L^p(G)$and $L^{p'}(G)$. In case the answer to both questions above is negative, we say that the group is *asymmetric.* In case the answer to the first question is affirmative, and to the second negative, we say the group is *weakly asymmetric*

Nilpotent groups Consider the Heisenberg group H of dimension 3, $H \cong \mathcal{R}^2 \times \mathcal{R}$ in which the multiplication law is

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy').$$

The group H is nilpotent and contains the subgroup $H_{\mathcal{Z}}$ consisting of elements with integer entries. If r is a prime, and $\mathcal{Z}_r$ is the field of r elements, then the group $H_{\mathcal{Z}}$ can be mapped in a canonical way homomorphically onto $H_{\mathcal{Z}_r}$ where $H_{\mathcal{Z}_r}$ is the analogue of the Heisenberg group in which the entries belong to the field $\mathcal{Z}_r$:

$$\pi_r : \qquad (x, y, z) \longrightarrow (x \bmod r, y \bmod r, z \bmod r).$$

This can be done for every large prime r. For large r, $H_{\mathcal{Z}_r}$ is rather like $H_{\mathcal{Z}}$. Furthermore, if T is a convolution operator on $L^p(H_{\mathcal{Z}_r})$, then π_r carries T in the obvious way into a convolution operator of the same norm on $L^p(H_{\mathcal{Z}})$.

These ideas led Herz to consider compact groups G which have an open normal subgroup H of index r. Starting with a *character* γ of H (i.e. a continuous homomorphism of H into the circle group $\mathcal{T}$), we may construct kernels k on $F \times H$ $(F = G/H)$ which are tensor products $k = \Phi \otimes \gamma$. If $\check{k}$ is the reflection of k $(\check{k}(g) = k(g^{-1}))$, then the norm of $T_{\check{k}}$ can be evaluated when $1 \leq p \leq 2$. Precisely,

$$||T_{\check{k}}||_p = ||\check{k}||_p \; r^{-1/p'}.$$

At the same time, the norm of $\check{k}$ can be evaluated by looking at its action on functions of the form $\Psi \otimes \gamma$. Using this idea, and playing ingeniously with the orthogonality of distinct characters of H, he was able to produce a kernel k and a number c_r which tends to zero as $r \longrightarrow \infty$ such that

$$||T_{\check{k}}||_4 \leq c_r \; r^{-1/4} \; ||\check{k}||_{4/3} = c_r \; ||T_{\check{k}}||_{4/3}.$$

Since r can be arbitrarily large, it follows that there exists on H a sequence $\{K_r\}_r$, r prime, of kernels such that

$$||T_{K_r}||_{4/3} = 1 \qquad (\forall r)$$

and

$$||T_{K_r}||_4 \longrightarrow 0$$

as $r \longrightarrow \infty$. Since every nilpotent nonabelian Lie group contains a copy of H, it follows that asymmetry pertains on every such group.

Solvable groups: the group $ax + b$ So far, no proof has been given of the asymmetry of solvable groups. I want to outline now some current work, joint with Michael Cowling, Anna Maria Mantero and Saverio Giulini, which aims to establish asymmetry for the most natural solvable group, viz. the affine group of the plane. This is the semi-direct product $\mathcal{A} = \mathcal{R}^* \odot \mathcal{R}$ in which

$$(a, b) \;\odot\; (a_0, b_0) = (aa_0, b + ab_0),$$

$a, a_0 \geq 0$, $b, b_0 \in \mathcal{R}$. The group $\mathcal{A}$ contains the subgroup $\mathcal{A}_0 = 2^{\mathcal{Z}} \odot \mathcal{R}$, which I will henceforth consider.

Let $K = \Phi \otimes \gamma$ be a kernel on $\mathcal{A}_0$, in which γ is a fixed 'nice' function. In particular, γ convolves $L^p(G)$into itself for every p. Then our calculations show that

(i)

$$||T_{\Phi \otimes \gamma}||_p = ||T_{(\check{\Phi} \otimes \check{\gamma})/\Delta}||_p$$

(ii) For every $p \in (1, \infty)$,

$$||T_{(\check{\Phi} \otimes \check{\gamma})/\Delta}||_p \;\geq\; ||T_\gamma||_{p,\mathcal{R}} \; ||\{\Phi(2^n)/2^n\}||_p$$

(iii)

$$||T_{(\check{\Phi} \otimes \check{\gamma})/\Delta}||_2 \leq \; ||\{\Phi(2^n)/2^n\}||_2 \; \sup_{r \geq 0} \Big(\sum_n \; |\hat{\gamma}(r2^n)|^2 \Big)^{1/2}$$

So, for instance, if $\hat{f}$ is of compact support disjoint from 0, then the last factor is finite.

(iv)

$$\begin{aligned} \|T_{(\check{\Phi} \otimes \check{\gamma})/\Delta}\|_1 &= \|\Phi \otimes \gamma\|_1 \\ &= \|\gamma\|_1 \ \|\{\Phi(2^n)/2^n\}\|_1 \end{aligned}$$

(v) By interpolation from (iii) and (iv),

$$\|T_{(\check{\Phi} \otimes \check{\gamma})/\Delta}\|_p \leq C_p(\gamma) \ \|\{\Phi(2^n)/2^n\}\|_p$$

when $1 \leq p \leq 2$.

(vi) We wish to know what

$$\|T_{(\check{\Phi} \otimes \check{\gamma})/\Delta}\|_{p'} = \|T_{(\Phi \otimes \gamma)/\Delta}\|_p$$

may be. Work is proceeding on constructing Φ so that $\|T_{(\Phi \otimes \gamma)/\Delta}\|_p$ remains bounded, while $\|\{\Phi(2^n)/2^n\}\|_p$ tends to $+\infty$.

HARDY-LITTLEWOOD MAXIMAL OPERATORS ON SOLVABLE GROUPS

In seeking to construct Φ with the desired properties, it is natural to consider some class of singular kernels on $\mathcal{A}_0$ or on $\mathcal{A}$ itself. In order to make progress in that direction, it is clearly desirable to have theorems about the boundedness of suitable maximal functions. I want to describe now some recent such results which are joint work of Saverio Giulini, Anna Maria Mantero, Andrej Hulanicki and myself.

Now $\mathcal{A}$ is in fact a symmetric space. It arises when one performs the Iwasawa decomposition of the group $SL_2(\mathbf{R})$:

$$SL_2(\mathbf{R}) = ANK$$

where

$$K = \left\{ \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \ : \ \theta \in [-2\pi, 2\pi) \right\}$$

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \ : \ t \in \mathbf{R} \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \ : \ c \in \mathbf{R} \right\}$$

The group $\mathcal{A} = AN$ is identifiable with the upper $\frac{1}{2}$-plane, and the action of an element of $\mathcal{A}$ on $G/K \cong AN$ is the same as the group multiplication on AN.

Now there is a G-invariant metric on AN given by $ds^2 = (dx^2 + dy^2)/y^2$, where AN is identified with $(0, +\infty) \times (-\infty, +\infty)$. This so-called hyperbolic structure gives a G-invariant metric and balls which are ordinary discs. With this structure, a natural neighbourhood base at $e \sim i$ is given by the family of discs $\{D_r\}$ where D_r has centre the point $(0, \cosh r)$, is of radius $\sinh r$, and the measure of D_r is asymptotically e^{2r} as $r \longrightarrow +\infty$. With this system of neighbourhoods, associate the maximal function

$$M_s f(g) = \sup_{r \geq 0} |gD_r|^{-1} \int_{gD_r} f(\zeta) \, d\zeta.$$

The following result is due to J.-O. Strömberg [4].

THEOREM. *The operator M_s is of weak type* $(1, 1)$.

Strömberg's result is in fact general, and applies to the G-invariant structure on an arbitrary noncompact symmetric space.

Consider now the following family of solvable groups, of which $ax+b$ is typical. Let N be a nilpotent, simply connected, connected Lie group, and let A be a commutative, connected d-dimensional Lie group of automorphisms of N which are semi-simple on the Lie algebra $\mathfrak{n}$ of N, with positive eigenvalues. Identify each of the groups A and N with its Lie algebra. Then for each $t \in A$, e^t is an automorphism of N and

$$(e^{t_1} \cdot e^{t_2})x = e^{t_1+t_2}x.$$

Consider the split extension of N by $\mathfrak{a}$: $S = N \odot \mathfrak{a}$, the multiplication being

$$(x, t)(x', t') = (x + e^{-t}x', t + t').$$

The left and right Haar measures on S are, respectively,

$$\begin{aligned} d\mu_l(x,t) &= e^{-Tr\ t}\ dxdt \\ d\mu_r(x,t) &= \ dxdt \end{aligned}$$

Suppose we now construct the following family of "rectangles" about the identity of S. Let $x \longrightarrow |x|$ be a continuous nonnegative function on N such that the measure of the set $\{x : |x| \leq r\}$ behaves asymptotically like r^Q as $r \to 0$, $+\infty$. For $t \in \mathfrak{a}$, let $|t|$ denote the operator norm of t acting on N. Then the corresponding family of rectangles about e is given by

$$\mathcal{B}_r = \{s = (x, t) :\ |x| \leq r, |t| \leq r\}.$$

There is the corresponding maximal function

$$\mathcal{M}f(g) = \sup_{r \geq 0} \mu_l(\mathcal{B}_r)^{-1} \int_{g\mathcal{B}_r} f\ d\mu_l.$$

We have the following theorem.

THEOREM. *The maximal function $\mathcal{M}$ is of weak type (1, 1).*

Remarks. (i) The proof of our Theorem proceeds by showing that (a) the maximal function taken relative to rectangles of parameter size less than 1 can be estimated by using standard covering lemma techniques. (b) The maximal function $\mathcal{M}_\infty$ taken relative to rectangles such that $r \geq 1$ is majorized by the operator $f \star \check{\tau}$ where

$$\tau(x,t) = C/\{1 + |x|^Q(\sinh|x|)^d + |t|^Q(\sinh|t|)^d\},$$

and τ is right Haar integrable.

(ii) This result is not derivable from the theorem of Strömberg since the hyperbolic balls and our rectangles are not comparable in measure.

(iii) The operator $\mathcal{M}$ commutes with left translations, and might therefore be useful for controlling operators defined by convolution with kernels on the *right.*

(iv) Further investigation has shown that, if instead of translating the rectangles $\mathcal{B}_r$ on the left, we translate them on the right, then the corresponding maximal operator is not of weak type (p, p) for any p. On the other hand, if we also invert the rectangles, then the corresponding maximal operator is of *strong* type (p,p) when $p > 1$.

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