INHOMOGENEITIES IN A FRIEDMAN UNIVERSE

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Abstract: One of the outstanding problems in cosmology is the presence of inhomogeneities, which are characterized by an anisotropic pressure and density distribution. Following a method developed by McVittie (1967, 1968) it has been possible to find time-dependent spherically symmetric solutions of Einstein's field equations containing an arbitrary pressure and density distribution which connect smoothly to a Friedman universe for any desired equation of state.

1. INTRODUCTION

The problem of inhomogeneities in relativity has been considered already extensively in the literature. The simplest problem of a static sphere of uniform density was solved already in 1916 in a classic paper by K. Schwarzschild, who obtained both the external and internal solutions. The case of a static sphere of non-uniform density was examined exhaustively by Wheeler et al (1964). The collapse of a pressure-free distribution of mass was studied by Oppenheimer and Snyder (1939). More general distributions were considered by Tolman (1934), but unless one limits oneself to pressure-less distributions or at most such where the pressure is only a function of the time the problem becomes not amenable to analytical solution. Numerical solutions of the general time dependent case have been given by May and White (1966). Uniform density distributions but general pressure distributions have been successfully solved by Thompson and Whitrow (1967).

The description of inhomogeneities in cosmology immediately raises the question of boundary conditions at the surface of these inhomogeneities. While the universe as a whole can be described by a Robertson-Walker line element which is isotropic and will contain only a time-dependent pressure and density distribution, this is not the case for the inhomogeneities, which at least are radial dependent as well. Unless one limits oneself to pressure-less dust, which seems physically unrealistic, this raises a problem of matching the pressure distribution at the boundary. (The corresponding problem of matching the density distribution can be overcome by assuming a discontinuity at the boundary, although one would prefer a smooth cross-over there as well). In addition, one would like to have a line element for the interior solution, whose gravitational potentials connect smoothly to the corresponding values outside the distribution.

Some time ago, McVittie (1967, 1968) investigated a large class of time-dependent spherically symmetric solutions of Einstein's field equations containing a non-vanishing pressure and density distribution which, moreover, have time as well as radial dependence. It is the purpose of this note to show that his solution can be adapted to our problem of finding interior solutions which connect smoothly to a Friedmann universe for *any* desired equation of state. It is shown that this can be achieved by a transformation of the time-coordinate. The local time of the interior solution is not the universal time defined by the exterior one, which is uniquely defined, since $g_{44} = 1$ for a Friedman universe. In addition to requiring that the potentials match at the boundary we demand that under this transformation the pressure of the interior distribution connects smoothly to the one of the exterior solution at the boundary. This results in a number of conditions and limits the permitted values of the constants occurring in McVittie's solutions. This, incidentally, also guarantees that the density distribution carries smoothly from the interior one to that pertaining to the exterior at the boundary, but in no way puts any limitations on a possible equation of state connecting the pressure and density distribution.

2. INTERIOR SOLUTION

The line element inside the material in comoving coordinates is given by

$$ds^2 = y^2 dt^2 - (R_0^2/c^2)e^h S^2(dr^2 + f^2(d\theta^2 + \sin^2\theta\, d\phi^2)) \tag{2.1}$$

where $h = h(z)$, $y = y(z)$ are dimensionless functions of the variable z, defined by $e^z = Q(r)/S(t)$, $Q = Q(r)$ and $f = f(r)$ are dimensionless functions of r, and $S = S(t)$ a function of the time t. From the comoving nature ($T^{14} = 0$) it follows that

$$y = 1 - \tfrac{1}{2}h_z \tag{2.2}$$

(as usual subscripts denote differentiation with respect to that variable).

Furthermore, the condition for an isotropic distribution

$$T_1^1 = T_2^2 = T_3^3$$

and considering r and z as independent variables results in three equations for the functions f, Q and y (McVittie 1967),

$$Q_{rr}/Q + Q_r f_r/Qf = a(Q_r/Q)^2 \tag{2.3}$$

$$f_{rr}/f - f_r^2/f^2 + 1/f^2 = b(Q_r/Q)^2 \tag{2.4}$$

$$y_{zz} + (a - 3 + y)y_z + y[a + b - 2 - (a-3)y - y^2] = 0 \tag{2.5}$$

where a and b are constants. If $b = 0$ the solution of (2.4) is given by

$$f = \sin r, \qquad f = r, \qquad f = \sinh r \tag{2.4a}$$

where in anticipation of further results the constants of integration have been set equal to 1 and 0 respectively. The three solutions, of course, correspond to a closed, flat or open universe respectively. At the boundary of the inhomogeneity $r = r_b$ we can set, without loss of generality, $Q(r_b) = Q_b = 1$ so that the solution of (2.3) is given by

$$Q^{1-a} = 1 + (1-a)A^{-1}(T_k - T_{kb}) \tag{2.3a}$$

where

$$T_1 = -\cos r, \qquad T_0 = \tfrac{1}{2}r^2, \qquad T_1 = \cosh r \tag{2.3b}$$

corresponding to the values of $k = 1, 0, -1$, T_{kb} denotes the value at the boundary $r = r_b$ and A is a constant. We now turn to (2.5) which is of the form of Abel's equation and is integrable in terms of elementary functions only in certain cases. For these the first integral is given by

$$y_z = m(y+p)(y+q) \tag{2.5a}$$

where only certain values of m, p and q (as well as a) are possible.*

Finally, integrating (2.5a) results in

$$y = (pK - qu)/(u - K) \tag{2.5b}$$

where

$$u = e^{-2mdz} = (S/Q)^{2md}, \qquad d = \tfrac{1}{2}(p - q), \qquad K = \text{const. of integration.}$$

Note that at the boundary

$$u_b = S^{2md}.$$

Inserting (2.5b) into (2.2) then gives

$$e^h = C^2(u - K)^{2/m} u^{-(p+1)/md} \tag{2.2a}$$

where C is also a constant of integration. Furthermore, the pressure p and density ρ are given respectively by

$$\begin{aligned} \frac{Kp}{c^2} = &- y^{-1}\{2S''/S + (3y - 2)(S'/S)^2 + \\ &(c^2/R_0^2)e^{-h}S^{-2}\big(yB_1 + 2(y^2 - y - y_z)B_2 + (1 - y)(y^2 - y - 2y_2)B_3\big)\} \end{aligned} \tag{2.6}$$

$$K\rho = 3(S'/S)^2 + (c^2/R_0^2)c^{-h}S^{-2}\big(3B_1 - 6(1-y)B_2 + 2y_z - (1 - y)(2a - 1 - y)B_3\big) \tag{2.7}$$

where we have set

$$B_1 = \frac{1 - f_r^2}{f^2}\,, \quad B_2 = \frac{f_r Q_r}{fQ}\,, \quad B_3 = (Q_r/Q)^2.$$

3. BOUNDARY CONDITIONS

We now require that at the boundary $r = r_b$ the solution described by (2.3a), (2.4a), (2.5b) and (2.2a) for the line element (2.1) go over into that corresponding to a Friedmann universe, which can be written in the form

$$ds^2 = dT^2 - (R_0^2/c^2)R^2(T)(dX^2 + F_k^2(d\theta^2 + \sin^2\theta\, d\phi^2)) \tag{3.1}$$

*McVittie (Loc. cit.) lists four cases of which only two will be seen to satisfy our boundary conditions.

where

$$F_1 = \sin X, \quad F_0 = X, \quad F_{-1} = \sinh X$$

corresponding to $k = 1, 0, -1$. More important is that the pressure p calculated at the boundary be the one obtained by solving the field equations for (3.1), viz.

$$p/c^2 = -2\ddot{R}/R - \dot{R}^2/R^2 - kc^2/R^2 \tag{3.2}$$

where a dot indicates differentiation with respect to T.

An obvious way to achieve this would be to require that

$$t = T, \quad y_b = 1, \quad (y_z)_b = 0, \quad (e^z)_b = 1$$

together with

$$(dr/f)_{r_b} = dX/F.$$

Upon comparing this (2.5) we note that either we would have $y = 1$ throughout or $Q_b = 0$, either of which leads to trivial or unphysical results. Another possibility would be to carry out a transformation of the time. Since $g_{44} \neq 1$ for the interior solution, its local time is not the universal time defined by the exterior solution. From (2.5b) it follows that

$$y_b = (pK - qx)/(x - K), \quad x = u_b = S^{2md}. \tag{3.3}$$

Hence, we shall set

$$y_b = dT/dt \quad \text{or} \quad T = \int y_b dt. \tag{3.4}$$

Comparison of (2.1) with (3.1) shows that the remaining conditions to be satisfied are

$$(S^2 e^h)_b = R^2(T), \qquad (dr/f)_b = dX/F.$$

The latter is automatically satisfied by identifying $X = r$, while the first leads to

$$R = C(x - K)^{1/m} S^{-p}. \tag{3.5}$$

The question which remains to be answered is whether, under the transformation (3.4) the pressure as given by (2.6) transforms into (3.2) and if any additional conditions are to be met in order to effect that transformation. We first note that the derivatives in (2.6) are given with respect to t, while those in (3.2) are with respect to T. With the help of (3.4) and using (2.2) we find

$$R = R'/y_b, \quad R'/R = (S'/S)y_b \quad \text{so that} \quad \dot{R}/R = S'/S$$

as well as

$$\ddot{R}/R = (S''/S - S'^2/S^2)y_b^{-1}$$

from which it follows immediately that the terms involving S''/S and $(S'/S)^2$ identically transform into the corresponding terms of (3.2). What remains are the terms multiplying $1/R^2$

$$\begin{aligned} kc^2/R = (c^2 e^{-h}/S^2 y)_b \{ y_b B_{1b} - 2[y_b^2 - y_b - (y_z)_b] B_{2b} \\ - (1 - y_b)[y_b^2 - y_b - 2(y_z)_b] B_{3b} \} \end{aligned} \tag{3.6}$$

where $B_{ib}(i = 1, 2, 3)$ denote the values of B_i at r_b. Noting that $B_1 = 1$ if $k = 1$ (and $B_1 = 0, -1$ if $k = 0, -1$) and using again (3.5) it is seen that the transformation is complete provided the last two terms on the right hand side of (3.6) vanish. This will result in a cubic equation in y_b (or x) and if we demand that all its coefficients vanish, we obtain the following possible values for the various constants:

(i) $m = \frac{1}{2}, \quad a = \frac{1}{2}, \quad p = -1, \quad q = 0, \quad B_{2b} = 0$

(ii) $m = \frac{1}{2}, \quad a = 3, \quad p = 1, \quad q = -1, \quad B_{2b} + B_{3b} = 0$

(ii') $m = \frac{1}{2}, \quad a = 3, \quad p = -1, \quad q = 1, \quad B_{2b} + B_{3b} = 0.$

Comparison with the four cases listed by McVittie shows that our cases are included in his first two.

Turning now to the density (2.7) it is readily seen that it also reduces at the boundary to the Friedmann value. It follows that for either of the above possibilities

$$K\rho = 3(S'/S)^2 + 3kc^2/R^2 \tag{3.7}$$

which is exactly the Friedmann value. Thus, our interior solution goes smoothly into the exterior Friedmann solution at the boundary with the pressure and density taking their corresponding values.

4. RESULTS

What remains to be done is to use the values of the various constants and to calculate the different functions for the three cases. We shall consider each case in turn (i) Inserting the appropriate values we have

$$y = K/(K-u), \qquad e^h = C^2(u-K)^4, \qquad u = (Q/S)^{\frac{1}{2}}$$

so that the line element will have the form

$$ds^2 = (K/(K-u))^2 dt^2 - C^2(R_0^2/c^2)S^2(u-K)^4(dr^2 + f^2(d\theta^2 + \sin^2\theta\, d\phi^2)). \tag{4.1}$$

The values of Q as well as B_i will be different for different values of k. For $k=1$ we obtain

$$Q = 4\cos^4(r/2), \quad B_1 = 1, \quad B_2 = -2\cos r/(1-\cos r)$$

where the boundary is determined by $\cos r_b = 0$ and we have taken $A = -\frac{1}{2}$. For $k=0$ we have $Q=1$, $B_2 = B_1 = 0$, $A^{-1} = 0$. For $k=-1$ the trigonometric functions are replaced by hyperbolic ones, so that

$$Q = 4\cosh^4(r/2), \quad B_1 = -1, \quad B_2 = 2\cosh r/(1+\cosh r)$$

the boundary being determined by $\cosh r_b = 0$ and $A = \frac{1}{2}$. We have not listed B_3 since it turns out that the factor multiplying it vanishes in this case. The pressure turns out to be

$$\begin{aligned} \frac{K\rho}{c^2} = &-\frac{(K-u)}{K}\left\{\frac{2S''}{S} + \frac{K+2u}{K-u}\left(\frac{S'}{S}\right)^2\right. \\ &\left. + \frac{c^2}{R_0^2}\frac{S^{-2}}{C^2(K-u)^4}\left[k\frac{K}{K-u} - \frac{uK}{(K-u)^2}B_2\right]\right\} \end{aligned} \tag{4.2}$$

while the density is given by

$$K\rho = 3\left(\frac{S'}{S}\right)^2 + \frac{c^2}{R_j}\frac{S^{-2}}{C^2(K-u)^4}\left[3k + 6\frac{u}{K-u}B_2\right]. \tag{4.3}$$

If $k = 0$ (flat space) the result simplifies considerably. The pressure is given by

$$\frac{Kp}{c^2} = -\frac{(K-x)}{K}\left[2\frac{S''}{S^2} + \frac{K+2x}{(K-x)}\left(\frac{S'}{S^2}\right)\right], \quad x = S^{-\frac{1}{2}} \tag{4.2a}$$

while the density is just

$$K\rho = 3(S'/S)^2. \tag{4.3a}$$

(ii) In this case we have the following functions

$$y = (u+K)/(u-K), \qquad e^h = C^2(u-K)^4 u^{-4}, \qquad u = S/Q$$

$$y = (K+u)/(K-u), \qquad e^h = C^2(u-K)^4, \qquad u = Q/S. \tag{ii$'$}$$

For both cases Q as well as B_i are the same, but of course take different values for various values of k.

For $k = 1$,

$$Q = M^{-\frac{1}{2}}, \quad B_1 = 1, \quad B_2 = \cos r/AM, \quad B_3 = \sin^2 r/A^2M^2$$

where

$$M = 1 - (2/A)(\cos r_b - \cos r), \quad A = -\sin^2 r_b/\cos r_b.$$

For $k = 0$ we have

$$Q = r_b/r, \qquad B_1 = 0, \qquad B_2 = -B_3 = 1/r^2.$$

Finally, if $k = -1$ we obtain

$$Q = N^{-\frac{1}{2}}, \quad B_1 = -1, \quad B_2 = \cosh r/A'N, \quad B_3 = \sinh^2 r/A'^2N^2$$

where

$$N = 1 - (2/A')(\cosh r - \cosh r_b) \quad \text{and} \quad A' = -\sinh^2 r_b/\cosh r_b.$$

However, the pressure and the density are different in the two sub-cases.

In the first instance we find for the pressure

$$(4.4)\qquad \begin{aligned} \frac{Kp}{c^2} = \frac{K-u}{K+u}\Bigg\{ & 2\frac{S''}{S} + \frac{u+5K}{u-K}\left(\frac{S'}{S}\right)^2 + \\ & \frac{c^2}{R_0^2}\frac{S^{-2}u^4}{c^2(u-K)^4}\left(k\frac{u+K}{u-K} + \frac{4u^2}{(u-K)^2}B_2 - \frac{4uK}{(u-K)^2}B_3\right)\Bigg\} \end{aligned}$$

and for the density

$$(4.5)\qquad K\rho = 3\left(\frac{S'}{S}\right)^2 + \frac{c^2}{R_j^2}\frac{u^4S^{-2}}{c^2(u-K)^4}\left[3k + \frac{12K}{u-K}(B_2+B_3)\right]$$

while the corresponding values in the second subcase are given by

$$(4.4a)\qquad \begin{aligned} \frac{Kp}{c^2} = \frac{u-K}{u+K}\Bigg\{ & 2\frac{S''}{S} + \frac{K+Su}{K-u}\left(\frac{S'}{S}\right)^2 + \\ & \frac{c^2}{R_0^2}\frac{S^{-2}}{(K-u)^4}\left(k\frac{K+u}{K-u} + \frac{4u^2}{(K-u)^2}B_2 - \frac{4uK}{(K-u)^2}B_3\right)\Bigg\} \end{aligned}$$

and

$$(4.5a)\qquad K\rho = 3\left(\frac{S'}{S}\right)^2 + \frac{c^2}{R_0^2}\frac{S^2}{c^2(K-u)^4}\left[3k + \frac{12u}{K-u}(B_2+B_3)\right].$$

Some simplification is achieved if $k = 0$, since then $u = Sr/r_b$ and $u = r_b/Sr$ respectively, while the density in both cases is again of the form (4.3a). The scale function S is still undetermined and one would have to impose additional conditions, such as a relation between the pressure and density, to determine it.

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