THE NORMS OF POWERS OF FUNCTIONS IN THE VOLTERRA ALGEBRA, II

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In this note we provide an example of a weight sequence $\{\omega_n\}_{n\geq 1}$ which satisfies (i) $\omega_n \geq 0$, (ii) $\omega_{n+m} \leq \omega_n\omega_m$, (iii) $\omega_n^{1/n} \to 0$, and (iv) $\omega_n^{1/n}$ is monotone decreasing, but for which there is no positive $\mu \in (L^1[0,1],*)$ with $\omega_n = \|\mu^n\|$ for every n. This answers the problem of [1], whereas, as detailed there, the example of [2] is for a different, albeit related, problem.

LEMMA. *If* $\mu \in (L^1[0,1],*)$ *is positive and non-nilpotent, then* $\dfrac{\|\mu^{2n}\|}{\|\mu^n\|^2} \to 0$ *as* $n \to \infty$.

Proof. It is shown in [1] that $\|\mu^n\|^{\frac{1}{n}}$ is monotone decreasing. Hence,

$$\frac{\|\mu^{n+1}\|}{\|\mu^n\|} = \frac{(\|\mu^{n+1}\|^{\frac{1}{n+1}})^{n+1}}{(\|\mu^n\|^{1/n})^n}$$

$$= \left[\frac{\|\mu^{n+1}\|^{\frac{1}{n+1}}}{\|\mu^n\|^{1/n}}\right]^n \cdot \|\mu^{n+1}\|^{\frac{1}{n+1}}$$

$$\leq \|\mu^{n+1}\|^{\frac{1}{n+1}} \to 0 \text{ as } n \to \infty,$$

that is, the sequence $(\|\mu^n\|)_{n=1}^{\infty}$ is regulated.

Now let $J = \{f \in L^1[0,1]: \lim_{n\to\infty} \dfrac{\|f*\mu^n\|}{\|\mu^n\|} = 0\}$. Then J is a closed ideal in $(L^1[0,1],*)$, [2]. Since μ is not nilpotent, $\inf(\text{supp}(\mu)) = 0$, and since $\mu \in J$ it follows that $J = L^1[0,1]$. Therefore, $\lim_{n\to\infty} \dfrac{\|f*\mu^n\|}{\|\mu^n\|} = 0$ for every $f \in L^1[0,1]$. If p is a probability measure with support contained in $(0,\frac{1}{2})$, then $\|p*\mu^n\| \geq \|\delta_{1/2}*\mu^n\|$ and so

$$\lim_{n\to\infty} \frac{\|\delta_{1/2}*\mu^n\|}{\|\mu^n\|} = 0.$$

For each n, write $\mu^n = \mu_1^{(n)} + \mu_2^{(n)}$, where $\mu_1^{(n)} = \mu^n\big|_{[0,1/2]}$ and $\mu_2^{(n)} = \mu^n\big|_{(1/2,1]}$. Then $\|\mu_1^{(n)}\| = \|\delta_{1/2}*\mu^n\|$ and $\mu^{2n} = (\mu_1^{(n)})^2 + 2\mu_1^{(n)}*\mu_2^{(n)}$. Therefore,

$$\frac{\|\mu^{2n}\|}{\|\mu^n\|^2} = \frac{\|(\mu_1^{(n)})^2\|+2\|\mu_1^{(n)}*\mu_2^{(n)}\|}{\|\mu^n\|^2}$$

$$\leq \frac{\|\delta_{1/2}*\mu_n\|^2+2\|\delta_{1/2}*\mu^n\|\|\mu_2^{(n)}\|}{\|\mu^n\|^2}$$

$$\leq \left[\frac{\|\delta_{1/2}*\mu^n\|}{\|\mu^n\|}\right]^2 + 2\,\frac{\|\delta_{1/2}*\mu^n\|}{\|\mu^n\|} \longrightarrow 0 \text{ as } n \to \infty.$$

Now let $\{M_n\}_{n=1}^{\infty}$ be any positive sequence which increases to ∞. Set $\omega_n = e^{-nM_k}$ if $4^{k-1} \leq n < 4^k$. Then $\{\omega_n\}_{n=1}^{\infty}$ satisfies (i)-(iv) but for each $k = 1,2,3,\ldots$

$$\omega_{2\cdot 4^k} = e^{-2\cdot 4^k M_{k+1}} = \left[e^{-4^k M_{k+1}}\right]^2 = (\omega_{4^k})^2,$$

and so $\dfrac{\omega_{2n}}{\omega_n^2} \not\to 0$.

REFERENCES

[1] G. R. Allan, An inequality involving product measures, in *Radical Banach algebras and automatic continuity*, 277-279, Lecture Notes in Mathematics 975, Springer-Verlag, Berlin and New York, 1983.

[2] G. A. Willis, The norms of powers of functions in the Volterra algebra, in *Radical Banach algebras and automatic continuity*, 280-281, Lecture Notes in Mathematics, 975, Springer-Verlag, Berlin and New York, 1983.

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