FUNCTION THEORY ON BANACH ALGEBRAS

R. Parvatham

First let me recall some notions and results in function theory on the complex plane $\mathbb{C}$. Then by adopting suitable means they are extended to Banach algebras.

Let D be a region (an open connected set) in $\mathbb{C}$ and let H(D) be the class of all functions holomorphic in D. In general, in this work the study of univalent functions is confined to the class of functions $S = \{f \in H(E) : f(0) = 0$ and f is univalent in $E\}$ where $E = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc in $\mathbb{C}$.

A domain D in $\mathbb{C}$ is said to be *convex* if the line joining any two points in D lies in D . A function $f \in S$ is said to be *convex* in E if f(E) is a convex set. Let K denote the collection of all convex functions in E . The analytic criteria for $f \in K$ is $\mathcal{R}e\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > 0$ in E .

A domain D in $\mathbb{C}$ is said to be *starshaped with respect to a point* $O \in D$ if the line joining any point $a \in D$ to O lies completely in D . It is obvious that any convex domain is starshaped with respect to each of its points. A function $f \in S$ is said to be *starlike* in E if f(E) is a starshaped domain with respect to the origin. Let S^* denote the collection of all starlike functions in E . Clearly we have $K \subseteq S^*$. The analytic criteria for $f \in S^*$ is $\mathcal{R}e\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ in E . Thus we have Alexander's theorem, namely: $f \in K$ if and only if $zf'(z) \in S^*$. Also $f \in S^*$ can be equivalently put as $(1-t)f(E) \subseteq f(E)$, for all $t \in I = [0,1]$. For details of the study of geometric function theory on the complex plane, the readers are referred to [1].

Recently another new class S_c^* of functions that are starlike with respect to conjugate points has been introduced by Thomas and El Rabha [5]. A function $f \in S$

is said to be *starlike with respect to conjugate points* if $\mathcal{Re}\{2zf'(z)[f(z)+\overline{f(\bar{z})}]^{-1}\} > 0$ in E. It is easy to verify that $g(z) = f(z) + \overline{f(\bar{z})})/2 \in S^*$ whenever $f \in S_c^*$.

With this background let me pass on to function theory on a Banach algebra.

Let R be a commutative Banach algebra over the complex numbers with identity (denoted by e) and let $\mathcal{M}$ be the space of all maximal ideals in R. Then $\mathcal{M}$ is a compact Hausdorff space where the topology is the weakest topology on $\mathcal{M}$ such that for each $x \in R$ the Gelfand transformation $\hat{x}$ of x is a continuous function on $\mathcal{M}$. Assume further that the Gelfand homomorphism $x \to \hat{x}$ of R into $C(\mathcal{M})$ is an isometry so that $\|x\| = \sup\{|\hat{x}(M)| : M \in \mathcal{M}\}$ for all $x \in R$. Under this assumption we may, and do, identify $x \in R$ with its Gelfand transform $\hat{x} \in C(\mathcal{M})$. Let $B = \{x \in R : \|x\| < 1\}$. If D is an open set in R, $F : D \to R$ is said to be *L-analytic in* D, [4], if for each $x \in D$ there exists $F'(x) \in R$ such that

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - xF'(x)\|}{\|h\|} = 0 .$$

If $F : B \to R$ is L-analytic in B with $F(0) = 0$, then for each $x \in B$, $F(x) = \sum_1^\infty a_n x^n$ where $a_n \in R$ and the series converges uniformly on $\{x \in R : \|x\| < \delta\}$ for each $\delta < 1$, [3]. If $F : B \to R$ is L-analytic in B and for each $y \in F(B)$ there is an open neighbourhood V of y on which F^{-1} exists and is L-analytic, then we say that F is *locally bi-analytic* in B. If F is univalent and locally bi-analytic in B, then F is said to be *bi-analytic* in B.

If F is L-analytic in B then for each $M \in \mathcal{M}$, there is an associated holomorphic function $F_M : E \to \mathbb{C}$ defined by $F_M(z) = F(ze)(M)$ for all $z \in E$. If $F(x) = \sum_1^\infty a_n x^n$ is L-analytic in B then $F(x)/x$ stands for the L-analytic function $\sum_1^\infty a_n x^{n-1}$ in B.

Now let the notion of a starlike mapping be extended to R as follows.

DEFINITION 1. A bi-analytic map $F : B \to R$ is said to be *starlike* in B if $F(0) = 0$ and $(1-t)F(B) \subseteq F(B)$, for all $t \in I$.

DEFINITION 2. $U : D \to R$ is said to have *positive real part* if U is L-analytic in D and $\mathcal{R}e(U(x)(M)) \geq 0$ in D for each $M \in \mathcal{M}$ and each $x \in D$.

If in addition $\mathcal{R}e U(x)(M) > 0$ for all $M \in \mathcal{M}$ and $x \in D$, we write $U \in \mathcal{P}(D)$, and if $D = B$, then $\mathcal{P}$ is written for $\mathcal{P}(B)$.

DEFINITION 3. A bi-analytic map $F : B \to R$ is said to be *convex* in B if $F(B)$ is a convex domain.

The following results of [2] give the relation between these notions in R and $\mathbb{C}$.

THEOREM 1. [2] *Let* $F(x) = \sum_{n=1}^{\infty} a_n x^n$ *be locally bi-analytic in* B. *Then* F *is starlike in* B *if and only if* $F_M(z) = \sum_{n-1}^{\infty} a_n(M) z^n$ *is starlike in* E *for all* $M \in \mathcal{M}$.

THEOREM 2. [2] *Let* $F(x) = \sum_{1}^{\infty} a_n x^n$ *be locally bi-analytic in* B. F *is convex in* B *if and only if* $F_M(z) = \sum_{1}^{\infty} a_n(M) z^n$ *is convex* in E *for each* $M \in \mathcal{M}$. *Thus Alexander's relation holds, namely* F *is convex in* B *if and only if* $\Phi(x) = xF'(x)$ *is starlike in* B.

Amongst other results, the proofs of these use the following lemma which we need below.

LEMMA 1. [2] *Let* U *have positive real part in* B.

(1) *If* $M \in \mathcal{M}$, *then*

$$\frac{1-\|x\|}{1+\|x\|} \mathcal{R}e U(0)(M) \leq \mathcal{R}e U(x)(M) \leq \frac{1+\|x\|}{1-\|x\|} \mathcal{R}e U(0)(M) \quad \textit{for all } x \in B,$$

and so

$\mathcal{Re}U(0)(M) > 0$ *if and only if* $\mathcal{Re}U(x)(M) > 0$ *for all* $x \in B$.

(2) $\mathcal{Re}U(0)(M) > 0$ *for all* $M \in \mathcal{M}$ *implies* $U(x)$ *is nonsingular for all* $x \in B$.

Now let us define a new class of mappings in B which is analogous to the class S_c^* in $\mathbb{C}$. For this we need the basic space to be a Banach *-algebra.

A Banach algebra R is called a Banach *-algebra if it has an involution; that is, there is given a mapping $x \to x^*$ of R into itself such that (i) $(x+y)^* = x^* + y^*$, (ii) $(\alpha x)^* = \bar{\alpha}x^*$, (iii) $(xy)^* = y^*x^*$, (iv) $x^{**} = x$. It follows that $0^* = 0$ and $e^* = e$. The element x^* is called the adjoint of x. Let R^* be a commutative Banach *-algebra with identity e and $\mathcal{M}$ be the space of all maximal ideals of R^*. We will assume that $R^* = C(\mathcal{M})$ with natural involution. This is equivalent to assuming the Gelfand transform is isometric and symmetric.

DEFINITION 4. Let $B^* = \{x \in R^* : \|x\| < 1\}$. Suppose $F : B^* \to R^*$ is bi-analytic in B^*. We say that F is *starlike with respect to adjoint elements* in B^* if $F(0) = 0$ and $x\,F'(x)\,U(x) = G(x)$, for each $x \in B^*$, where $G(x) = (F(x) + F(x^*))^*)/2$ and U has positive real part in B^*.

THEOREM 3. *Let* $F(x) = \sum_{n=1}^{\infty} a_n x^n$ *be locally bi-analytic in* B^*. *Then* F *is starlike with respect to adjoint elements in* B^* *if and only if* $F_M(z) = \sum_{n=1}^{\infty} a_n(M)z^n$ *is starlike with respect to conjugate points in* E *for all* $M \in \mathcal{M}$.

PROOF. Assume that F is starlike with respect to adjoint elements in B^*. Then $x\,F'(x)\,U(x) = G(x) = \frac{1}{2}\,(F(x) + F(x^*))^*)$ where U has positive real part in B^*. However, by equating the coefficients, $\mathcal{Re}U(0)(M) = 1$ for all $M \in \mathcal{M}$ and hence $U \in \mathcal{P}$ by Lemma 1. Setting $x = ze$, and using the fact that $a_n^*(M) = \overline{a_n(M)}$, we conclude that

$$0 < \mathcal{R}e\,U(ze)(M) = \mathcal{R}e\left\{\frac{\sum_1^\infty (a_n(M) + \overline{a_n(M)})z^n}{2\sum_1^\infty na_n(M)z^n}\right\} \text{ in } E .$$

Thus F_M is starlike with respect to conjugate points in E.

Conversely assume that F_M is starlike with respect to conjugate points in E for every $M \in \mathcal{M}$. Set $P(\omega) = \frac{F(\omega) + (F(\omega^*))^*}{2\omega F'(\omega)}$ for all $\omega \in B^*$. Then $F_M \in S_c^*$ which implies $P \in \mathcal{P}$. Now to see the univalence of F, let $x_1, x_2 \in B^*$, $x_1 \neq x_2$, and choose $M \in \mathcal{M}$ so that $|(x_2-x_1)(M)| = \|x_2-x_1\|$. Note that whenever $F_M \in S_c^*$,

$$G_M(z) = \frac{F_M(z) + \overline{F_M(\bar{z})}}{2} = \sum_1^\infty \frac{a_n(M) + \overline{a_n(M)}}{2} z^n \in S^* .$$ Define $G : B^* \to R^*$ by

$$G(x) = \sum_1^\infty \frac{a_n(M) + \overline{a_n(M)}}{2} x^n .$$ Thus $G(x) = \frac{F(x) + (F(x^*))^*}{2}$, and by Theorem 1 G is starlike in B^* since $G_M(z) \in S^*$. By Alexander's relation there is a mapping Φ, convex in B^*, such that $G(x) = x\Phi'(x)$ for all $x \in B^*$; in particular Φ is bi-analytic. Now $F_M \in S_c^*$ implies $\mathcal{R}e\left\{z\frac{F_M'(z)}{G_M(z)}\right\} = \mathcal{R}e\left\{\frac{F_M'(z)}{\Phi_M'(z)}\right\} > 0$ in E.

Consider $H = F \circ \Phi^{-1} : \Phi(B^*) \to R^*$ and let $y_1 = \Phi(x_1)$ and $y_2 = \Phi(x_2)$. Then since $\Phi(B^*)$ is convex, $\{ty_2 + (1-t)y_1 : t \in I\} \subseteq \Phi(B^*)$, and so $F(x_2) - F(x_1) = H(y_2) - H(y_1) = \int_0^1 H'(ty_2+(1-t)y_1)(y_2-y_1)dt$. Thus

$$\begin{aligned}
|(F(x_2) - F(x_1))(M)| &= |(H(y_2) - H(y_1))(M)| \\
&= |(y_2-y_1)(M)| \cdot \left|\int_0^1 H'(ty_2 + (1-t)y_1)(M)dt\right| , \\
&\geq |(y_2-y_1)(M)| \cdot \int_0^1 \mathcal{R}e\,H'(ty_2+(1-t)y_1)(M)dt , \\
&= |(y_2-y_1)(M)| \cdot \int_0^1 \mathcal{R}e\left\{\frac{F_M'(\Phi^{-1}(ty_2+(1-t)y_1)(M))}{\Phi_M'(\Phi^{-1}(ty_2+(1-t)y_1)(M))}\right\} dt \\
&> 0
\end{aligned}$$

if $|(y_2-y_1)(M)| \neq 0$. But

$$0 \neq \|x_2-x_1\| = |(x_2-x_1)(M)| = |(y_2-y_1)(M)| \cdot \left| \int_0^1 (\Phi^{-1})'(ty_2+(1-t)y_1)(M)dt \right| ;$$

and hence $|(y_2-y_1)(M)| \neq 0$ which implies $F(x_2) \neq F(x_1)$, thereby giving the desired result.

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Ramanujan Institute for Advanced Study in Mathematics
University of Madras
Madras 600005
India.