INTERTWINING WITH ISOMETRIES

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This lecture contains work done jointly with P. Vrbová [6] and will develop some variations on a theme which goes back to Johnson and Sinclair [2].

The general question under scrutiny is that of continuity of intertwiners: if S, T are given linear operators on the vector spaces X, Y, respectively, then we consider the space

$$\mathcal{I}(S,T) := \{\theta : X \to Y \mid \theta \text{ linear}, \ C(S,T)^n\theta = 0 \ , \text{ some } n \in \mathbb{N}\} \ ,$$

where $C(S,T)^n$ is the nth composition of the map

$$C(S,T) \, : \, \theta \to S\theta - \theta T \ ,$$

and try to decide when $\mathcal{I}(S,T)$ consists of continuous maps (provided X, Y are Banach spaces and S, T are continuous).

The interest in the space $\mathcal{I}(S,T)$ stems from the fact that it contains many significant classes of maps:

If A, B are Banach algebras and $\theta : A \to B$ is an algebra homomorphism then $\theta \in \mathcal{I}\big(\theta(a), a\big)$ for any $a \in A$ in the sense that

$$\theta(a)\theta(x) - \theta(ax) = 0$$

for all $x \in A$.

Another class of examples emerges if X is a Banach algebra and Y is a commutative Banach X-module; if $D : X \to Y$ is a module derivation then $C(a,a)^2 D = 0$, as an easy calculation will show.

To state the main results we will need a few facts about the algebraic spectral subspaces $E_S(A)$ of a linear operator S on a vector space Y: given a subset $A \subseteq \mathbb{C}$,

$E_S(A)$ is the maximal subspace of Y among all subspaces Z for which

$$(S-\lambda)Z = Z \qquad \text{for all } \lambda \notin A\,.$$

In particular $E_S(\emptyset)$ is the largest S-divisible subspace of Y. It is clear that

$$E_S(A) \subseteq \cap_{\lambda\notin A,\, n\in\mathbb{N}}(S-\lambda)^n Y$$

and in some cases, e.g. when S is a generalized scalar operator, we have equality [5]. The only instance we need here is the case when $A = \mathbb{C}\backslash\{0\}$ and S is 1–1:

$$\text{If } S \text{ is 1–1 then} \quad E_S(\mathbb{C}\backslash\{0\}) = \bigcap_{n=1}^{\infty} S^n Y\,.$$

[Proof. If $y \in \cap S^n Y$ and $y = S^n y_n$, $n = 1,2,\ldots$ then $y_1 = Sy_2 = S^2 y_3 = \cdots$ so $y_1 \in \cap S^n Y$.]

Once we recall that $\lambda \in \mathbb{C}$ is a *critical* eigenvalue for (S,T) provided λ is an eigenvalue for S and $(T-\lambda)X$ is of infinite codimension in X, we are able to understand the statement of

THEOREM A. *If there is a countable set $G \subseteq \mathbb{C}$ for which $E_S(\mathbb{C}\backslash G) = \{0\}$ then every $\theta \in \mathcal{I}(S,T)$ is continuous if and only if (S,T) has no critical eigenvalues in G.*

Sketch of Proof. The eigenvalue condition is easily seen to be necessary. We indicate the line of attack in proving sufficiency. To simplify slightly, suppose $S\theta = \theta T$. First, by the stability lemma there is a polynomial p with roots in G for which

$$\big((S-\lambda)p(S)\mathfrak{S}\big)^- = \big(p(S)\mathfrak{S}\big)^-$$

for all $\lambda \in G$, where

$$\mathfrak{S} = \{y \in Y \mid \exists x_n \to 0 \quad \text{with } \theta x_n \to y\}\,.$$

[Actually, this can be done so as to hold for all $\lambda \in \mathbb{C}$, so countability of G does not really play a role here.] Second, by Mittag-Leffler's theorem there is a dense subspace $W \subseteq p(S)\mathfrak{S}$ for which

$$(S-\lambda)W = W\,, \qquad \lambda \in G\,.$$

[This *does* depend on countability of G.] By maximality of $E_S(\mathbb{C}\backslash G)$,

$$W \subseteq E_S(\mathbb{C}\backslash G)\,,$$

hence W, and thereby $p(S)\mathfrak{S}$, is $\{0\}$. Discard all non-eigenvalue roots of p. Thus $p(T)X$ is of finite codimension, by our assumption of no critical eigenvalues. From this the continuity of θ on all of X follows readily.

COROLLARY 1. *If S has countable spectrum and $E_S(\emptyset) = \{0\}$ then every $\theta \in \mathcal{I}(S,T)$ is continuous if and only if (S,T) has no critical eigenvalue.*

Proof. $G = \sigma(T)$ and

$$\begin{aligned} E_S(\mathbb{C}\backslash G) &= E_S\big((\mathbb{C}\backslash\sigma(T)) \cap (\sigma(T)\big) \\ &= E_S(\emptyset) = \{0\}\,. \end{aligned}$$

This is a good part of the original Johnson–Sinclair result.

An isometry S for which $\cap S^n Y = \{0\}$ is called a *semi-shift.* An obvious example is the unilateral right shift.

COROLLARY 2. *If $T \in B(X)$ is arbitrary and $S \in B(Y)$ is a semishift then $\mathcal{I}(S,T)$ consists of continuous maps.*

Proof. $E_S(\mathbb{C}\backslash\{0\}) = \{0\}$ and 0 is not an eigenvalue of an isometry.

We shall now extend this last result to arbitrary isometries S and thus work our way away from the countability condition necessitated by our use of Mittag-Leffler. The cost of this is a mild restriction on T, namely the assumption that T be decomposable.

Thanks to Ernst Albrecht, [1], we may define T to be decomposable provided for any open cover $U \cup V = \mathbb{C}$ there are closed T-invariant subspaces X_U, X_V for which

$$\sigma(T \mid X_U) \subseteq U\,, \quad \sigma(T \mid X_V) \subseteq V$$

and

$$X = X_U + X_V\,.$$

THEOREM B. *If T is a decomposable map on the Banach space X and S is an isometry on the Banach space Y then $\mathcal{I}(S,T)$ consists entirely of continuous maps if and only if (S,T) has no critical eigenvalue.*

The main step in proving B is contained in

PROPOSITION C. *Suppose $S \in B(Y)$ is bounded below and satisfies $\cap S^n Y = \{0\}$, and suppose $T \in B(X)$ is decomposable. Then $\mathcal{I}(S,T) = \{0\}$.*

Proof. As in the proof of Corollary 2, $\mathcal{I}(S,T)$ consists of continuous maps. So suppose $\theta \in \mathcal{I}(S,T)$. To omit some of the technical details, suppose also that $S\theta = \theta T$. We know that $\ker\theta$ is closed and as $T(\ker\theta) \subseteq \ker\theta$ we may consider $\widetilde{T} : X/\ker\theta \to X/\ker T$ defined by $\widetilde{T}(x + \ker\theta) := Tx + \ker\theta$. If we let $\theta_1 : X/\ker\theta \to Y$ be defined by $\theta_1(x + \ker\theta) := \theta x$, then $\theta_1 \in \mathcal{I}(S, \widetilde{T})$.

But $\widetilde{T}$ is quasi-nilpotent: let $\varepsilon \in \mathbb{R}_+$ and cover $\mathbf{C}$: $\mathbf{C} = \{|z| < \varepsilon\} \cup (\mathbf{C}\backslash\{0\}) = U \cup V$. Then by decomposability we obtain a splitting

$$X = X_U + X_V .$$

Since $\sigma(T \mid X_V) \subseteq V$ we see that

$$X_V \subseteq E_T(\mathbf{C}\backslash\{0\}) .$$

Moreover, since

$$\theta E_T(\mathbf{C}\backslash\{0\}) \subseteq E_S(\mathbf{C}\backslash\{0\}) = \{0\}$$

(the inclusion is a consequence of the maximality of $E_S(\mathbf{C}\backslash\{0\})$, and since

$$S\theta E_T(\mathbf{C}\backslash\{0\}) = \theta T E_T(\mathbf{C}\backslash\{0\}) = \theta E_T(\mathbf{C}\backslash\{0\}) ,$$

we get that $X_V \subseteq \ker\theta$, and hence that $X = X_U + \ker\theta$. From this it follows that $\sigma(\widetilde{T}) \subseteq U$ and the arbitrariness of ε shows that $\sigma(\widetilde{T}) = \{0\}$.

Suppose $S^{-1} : \operatorname{ran} S \to Y$ has norm $\|S^{-1}\| = \delta_0$. This means that

$$\|Sy\| \geq \frac{1}{\delta_0}\|y\| \quad \text{for every } y \in Y$$

and hence

$$\|S^n y\| \geq \frac{1}{\delta_0^n}\|y\| \quad \text{for every } n\,.$$

Choose $n_0 \in \mathbb{N}$ so that

$$\|\widetilde{T}^{n_0}\| < \left(\frac{1}{2\delta_0}\right)^{n_0}$$

and note that

$$\begin{aligned}\|\theta_1 X\| &\leq \delta_0^{n_0}\|S^{n_0}\theta_1 x\| = \delta_0^{n_0}\|\theta_1 \widetilde{T}^{n_0} x\| \\ &\leq \delta_0^{n_0}\|\theta_1\|\,\|\widetilde{T}^{n_0}\|\,\|x\| \leq \left(\frac{\delta_0}{2\delta_0}\right)^{n_0}\|\theta_1\|\,\|x\| \\ &= 2^{-n_0}\|\theta_1\|\,\|x\|\end{aligned}$$

from which we get the unlikely claim that

$$\|\theta_1\| \leq 2^{-n_0}\|\theta_1\|\,,$$

which is only possible with $\theta_1 = 0$, hence $\theta = 0$.

Now the proof of B is not difficult.

Proof of B. With $Z := E_S(\mathbf{C}\backslash\{0\}) = \cap S^n Y$, Z is closed and S-invariant. If we let S induce $S_0 : Y/Z \to Y/Z$ then S_0 is a semi-shift. Moreover, if $Q : Y \to Y/Z$ is the quotient map then $Q\theta \in \mathcal{I}(S_0, T)$. Hence, by Proposition C, $Q\theta = 0$ so that θ maps X into Z. However, $S \mid Z$ is an invertible isometry ($S \mid Z$ is 1–1 and onto Z) so $\sigma(S \mid Z)$ is a subset of the unit circle $\mathbf{T}$. This means that $S \mid Z$ has a functional calculus (given by $C^\infty(\mathbf{C}) \ni f \to f \mid \mathbf{T} \to$ Fourier coefficients $(c_n(f))$ of $f \mid \mathbf{T} \to \sum_{n\in\mathbf{Z}} c_n(f)T^n$), so that $S \mid Z$ is generalized scalar. Since $(S \mid Z, T)$ has no critical eigenvalues, if (S, T) has no critical eigenvalues, the sufficiency of the critical eigenvalue condition follows from known results [3,4,7].

REFERENCES

[1] E. Albrecht, On decomposable operators, *Integral Equations Operator Theory* **2**(1979), 1–10.

[2] B.E. Johnson and A.M. Sinclair, Continuity of linear operators commuting with linear operators II, *Trans. Amer. Math. Soc.* **146**(1969), 533–540.

[3] K.B. Laursen, Algebraic spectral subspaces and automatic continuity, *Czechoslovak Math. J.* (113)**38**(1988), 157–172.

[4] K.B. Laursen and M.M. Neumann, Decomposable operators and automatic continuity, *J. Operator Theory* **15**(1986), 33–51.

[5] K.B. Laursen and P. Vrbová, Some remarks on the surjectivity spectrum of linear operators, submitted to Czechoslovak Math. J..

[6] K.B. Laursen and P. Vrbová, Intertwiners and automatic continuity, in preparation.

[7] P. Vrbová, On continuity of linear transformations commuting with generalized scalar operators in Banach space. *Čas. pěs. mat.* **97**(1972), 142–150.

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