BANACH ALGEBRA TECHNIQUES AND EXTENSIONS OF OPERATOR-VALUED REPRESENTATIONS

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1. INTRODUCTION

The existence of a functional calculus associated to a (bounded linear) operator T on a (complex) Banach space E can be very useful in the study of T, provided this functional calculus is defined on a sufficiently rich class of functions. In this note we consider several situations where standard Banach algebra techniques (mainly the use of a bounded approximate identity via Cohen's factorization theorem for modules) lead to extensions of a given functional calculus to a larger algebra. The typical case we discuss (§3) is that of a representation Φ of the standard disc algebra $\mathcal{A}(\bar{\mathbf{D}})$ into the Banach algebra $\mathcal{L}(E)$ of operators on E. (Indeed, any contraction on a Hilbert space gives rise to such a representation via von Neumann's inequality.) In this situation we can extend Φ to subalgebras H^∞_Γ of H^∞ (see below for definitions) where Γ is an open subset of the unit circle whose complement in $\mathbf{T}$ has (Lebesgue) measure 0. It turns out that such algebras have already been considered in the literature (cf. [3]). We conclude this section with an investigation of the "maximal" extension that can be obtained in this fashion.

In section 4 we discuss the same problem where the disc algebra is replaced by an arbitrary function algebra $\mathcal{A}$. Particular cases of this situation had already been studied in [1]. Here the "leading thread" is the connection between peak sets of $\mathcal{A}$ and bounded approximate identities for certain ideals of $\mathcal{A}$.

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2. PRELIMINARIES

For the reader's convenience, before stating the main extension tool, we recall the basic facts on multipliers and bounded approximate identities in a commutative Banach algebra $\mathcal{A}$ such that $\mathcal{A}^{\perp} = 0$ (that is, for all $f \in \mathcal{A}\backslash\{0\}$, there is $g \in \mathcal{A}$ with $fg \neq 0$). This material is essentially taken from [1] and [4].

DEFINITION 2.1. A *multiplier* of $\mathcal{A}$ is an operator $T : \mathcal{A} \to \mathcal{A}$ such that

$$T(g) = f\,Tg \quad (f, g \in \mathcal{A}).$$

The set $\mathcal{M}(\mathcal{A})$ of all multipliers of $\mathcal{A}$ turns out to be a norm-closed commutative subalgebra of $\mathcal{L}(\mathcal{A})$. The topology induced on $\mathcal{M}(\mathcal{A})$ by the strong operator topology of $\mathcal{L}(\mathcal{A})$ is referred to as the strong topology. There is a natural contractive embedding of $\mathcal{A}$ into $\mathcal{M}(\mathcal{A})$, $f \to T_f$ (where $T_f(g) = fg$, $g \in \mathcal{A}$). In general this embedding need not be bounded below. However it will in the case when $\mathcal{A}$ has a so-called bounded approximate identity.

DEFINITION 2.2. A *bounded approximate identity* for $\mathcal{A}$ (notation b.a.i.) is a bounded net $(e_\alpha)_{\alpha\in\Lambda}$ in $\mathcal{A}$ such that $\lim_\alpha \|fe_\alpha - f\| = 0$ for all f in $\mathcal{A}$.

The above assertion follows immediately from the definition: if the net (e_α) is bounded by M then we have $\|f\| \leq M\|T_f\|$ for all f in $\mathcal{A}$. Thus, when $\mathcal{A}$ has a b.a.i. we may consider $\mathcal{A}$ as a subalgebra of $\mathcal{M}(\mathcal{A})$ (this identification is in fact isometric if the b.a.i. is bounded by 1). If the net $(e_\alpha)_{\alpha\in\Lambda}$ can be taken to be a sequence then we say that $\mathcal{A}$ has a *sequential bounded approximate identity* (s.b.a.i.). It is well-known (and an easy consequence of Cohen's factorization theorem for modules over Banach algebras with b.a.i., cf. [7, Theorem 32.23, p.268]) that a Banach algebra $\mathcal{A}$ with b.a.i. has a s.b.a.i. if and only if $(a\mathcal{A})^- = \mathcal{A}$ for some $a \in \mathcal{A}$. In particular every separable Banach algebra with b.a.i. has a s.b.a.i.

We can now state the basic tool for extending representations. For completeness we include the proof. Here, as usual, by a representation we mean a norm-continuous algebra homomorphism.

THEOREM 2.3. (cf. [4, Prop. 5.2] and [1, Theorem 4.6]) *Let $\mathcal{A}$ be a commutative Banach algebra (satisfying $\mathcal{A}^{\perp} = 0$) with* s.b.a.i. *(bounded by* 1*) and let Φ be a representation of $\mathcal{A}$ into $\mathcal{L}(E)$ such that $\Phi(\mathcal{A})\, E$ is dense in E. Then*

(1) *Φ has a unique homomorphic extension $\tilde{\bar{\Phi}}$ from $\mathcal{M}(\mathcal{A})$ into $\mathcal{L}(E)$, and*

(2) *$\tilde{\bar{\Phi}}$ is norm continuous; in fact it is strong operator continuous and $\|\tilde{\bar{\Phi}}\| = \|\Phi\|$.*

Remark. The condition that the s.b.a.i. be bounded by 1 is no serious restriction since (cf. [8, Theorem 8]) any Banach algebra with s.b.a.i. can be (equivalently!) renormed so as to have a s.b.a.i. bounded by 1.

Proof of Theorem 2.3. (1) The representation Φ induces an $\mathcal{A}$-module structure on E via the operation $(a, x) \mapsto \Phi(a)x$. Since $\mathcal{A}$ has a s.b.a.i., it follows by Cohen's factorization theorem for modules (cf. [7, p.268]) that $\Phi(\mathcal{A})\, E$ is closed. Let $x \in E$; then $x = \Phi(a)\, y$ for some $a \in \mathcal{A}$ and $y \in E$. If $\tilde{\bar{\Phi}}$ is any homomorphic extension of Φ to $\mathcal{M}(\mathcal{A})$ then for any T in $\mathcal{M}(\mathcal{A})$ we have

$$\tilde{\bar{\Phi}}(T)\, x = \tilde{\bar{\Phi}}(T)\, \Phi(a)\, y = \tilde{\bar{\Phi}}(Ta) \;= \Phi(Ta)\, y\,.$$

This proves the uniqueness of $\tilde{\bar{\Phi}}$; moreover we have

$$\tilde{\bar{\Phi}}(T)\, x = \Phi(Ta)\, y = \lim_{n\to\infty} \Phi(e_n\, Ta)\, y = \lim_{n\to\infty} \Phi(Te_n)\, \Phi(a)\, y$$

(where $(e_n)_{n\in\mathbf{N}}$ is a s.b.a.i. bounded by 1), that is,

$$\tilde{\bar{\Phi}}(T)\, x = \lim_{n\to\infty} \Phi(Te_n)\, x\,.$$

It is now straightforward to check that $\tilde{\bar{\Phi}}$ defined by this last equality is indeed an algebra homomorphism.

(2) From the definition of $\tilde{\bar{\Phi}}$ we get immediately

$$\|\tilde{\bar{\Phi}}(T)\, x\| \le \limsup_{n\to\infty} \;\|\Phi(Te_n)\|\, \|x\| \le \|\Phi\|\, \|T\|\, \|x\| \quad (x \in E,\ T \in \mathcal{M}(\mathcal{A}))\,.$$

Thus $\tilde{\bar{\Phi}}$ is norm continuous and $\|\tilde{\bar{\Phi}}\| = \|\Phi\|$, (if (e_n) were bounded by M instead of 1 then we would get $\|\tilde{\bar{\Phi}}\| \le M\|\Phi\|$).

Let now $(T_\alpha)_{\alpha\in\Lambda}$ be a net in $\mathcal{M}(\mathcal{A})$ converging strongly to 0. Then (with $x = \Phi(a)\,y$)

$$\lim_\alpha \tilde{\Phi}(T_\alpha)\,x = \lim_\alpha \Phi(T_\alpha\,a)\,y = 0\,.$$

This completes the proof of (2).

3. THE DISC ALGEBRA CASE

In this section Φ is a given unital representation of the disc algebra $\mathcal{A}(\bar{\mathbf{D}})$ into $\mathcal{L}(E)$ such that $\Phi(\alpha) = T$ where α is the position function $z \mapsto z$. We will follow the standard practice of writing $f(T)$ for $\Phi(f)$ or $\tilde{\Phi}(f)$ when $\tilde{\Phi}$ is an extension of Φ and f belongs to the domain of definition of $\tilde{\Phi}$. Of course, since $\mathcal{A}(\bar{\mathbf{D}})$ is unital, $\mathcal{M}(\mathcal{A}(\bar{\mathbf{D}})) = \mathcal{A}(\bar{\mathbf{D}})$; thus we have to apply Theorem 2.3 to suitable nonunital subalgebras of $\mathcal{A}(\bar{\mathbf{D}})$ in order to obtain a nontrivial extension. A basic example is the ideal $\mathcal{A}_1(\bar{\mathbf{D}})$ of those functions in $\mathcal{A}(\bar{\mathbf{D}})$ vanishing at 1. It is immediate to check that the sequence of functions (e_n), $e_n(z) = (1-z)^{1/n}$, $z \in \bar{\mathbf{D}}$, is a s.b.a.i. in $\mathcal{A}_1(\bar{\mathbf{D}})$. Moreover the algebra $\mathcal{M}(\mathcal{A}_1(\bar{\mathbf{D}}))$ can be identified with the algebra $\mathcal{A}_1^\infty(\bar{\mathbf{D}})$ consisting of the functions bounded and analytic in $\mathbf{D}$ (the open unit disc) and continuously extendable to $\bar{\mathbf{D}}\backslash\{1\}$.

Finally we observe that the condition "$\Phi(\mathcal{A}_1(\bar{\mathbf{D}}))\,E$ dense in E" is obviously satisfied whenever $((1-T)\,E)^- = E$. Thus, from Theorem 2.3 we deduce immediately the following result.

THEOREM 3.1. *Let Φ be a representation from $\mathcal{A}(\bar{\mathbf{D}})$ into $\mathcal{L}(E)$ such that* $\mathrm{Ran}(1-T)$ *is dense in E. Then Φ has a unique homomorphic extension to the algebra $\mathcal{A}_1^\infty(\bar{\mathbf{D}})$. Moreover this extension is norm-continuous with the same norm as Φ.*

Of course we have a similar result, that is, an extension of Φ to

$$\mathcal{A}_\lambda^\infty(\bar{\mathbf{D}}) = \{f \text{ bounded, analytic on } \mathbf{D} \text{ and continuously extendable to } \bar{\mathbf{D}}\backslash\{\lambda\}\}\,,$$

for any point λ of $\mathbf{T}$ such that $(T-\lambda)$ has dense range.

Before turning our attention to another extension we state a "spectral mapping theorem" whose proof is another illustration of the usefulness of b.a.i. This result was initially proved, in the case where T is a completely nonunitary contraction on a Hilbert space, in [5], via entirely different techniques. Here as usual H^∞ denotes the Banach algebra of bounded analytic functions in $\mathbf{D}$ and, for $A \in \mathcal{L}(E)$, $\sigma(A)$ denotes the spectrum of A in $\mathcal{L}(E)$.

THEOREM 3.2. (cf. [1, Theorem 2.6]). *Let $\mathcal{A}$ be a subalgebra of H^∞ containing $\mathcal{A}(\bar{\mathbf{D}})$, let Φ be a representation of $\mathcal{A}$ into $\mathcal{L}(E)$. Then, for any $\lambda \in \sigma(T) \cap \mathbf{T}$ and any h in $\mathcal{A}$ continuously extendable to $\mathbf{D} \cup \{\lambda\}$, $h(\lambda)$ belongs to $\sigma(h(T))$. (As before $T = \Phi(\alpha)$ and $h(T) = \Phi(h)$.)*

Proof. By a "rotation" we may assume $\lambda = 1$. Then for h in H^∞ continuously extendable to $\mathbf{D} \cup \{1\}$ we have

$$\lim_{n\to\infty} (h - h(1))\, e_n = h - h(1)\,,$$

where, as before, $e_n(z) = (1-z)^{1/n}$. If in addition h belongs to $\mathcal{A}$, we have therefore

$$h(T) - h(1) = \lim_{n\to\infty} ((h - h(1))\, e_n)(T)\,.$$

If $h(1) \notin \sigma(h(T))$ then, since the set of invertible elements is open, we have, for n large enough, $(h(T) - h(1))\, e_n(T)$ invertible and consequently $e_n(T)$ invertible. This is absurd because, as is well-known and easily proved, for any f in $\mathcal{A}(\bar{\mathbf{D}})$ we have $f(\sigma(t)) \subseteq \sigma(f(T))$ and hence 0 ($= e_n(1)$) belongs to $\sigma(e_n(T))$. Thus $(h(T) - h(1))$ is not invertible, as was to be proved.

A more general example of extension is obtained by taking a closed subset F of $\mathbf{T}$ of Lebesgue measure 0 and replacing $\mathcal{A}_1(\bar{\mathbf{D}})$ by the ideal $\mathcal{A}_F(\bar{\mathbf{D}}) = \{f \in \mathcal{A}(\bar{\mathbf{D}}) \,:\, f|_F = 0\}$. By [6, II.12.6] there exists a function $p_F \in \mathcal{A}(\bar{\mathbf{D}})$ "peaking" on F, that is:

$$\begin{cases} p_F|_F = 1 \\ |p_F(z)| < 1, \quad z \in \bar{\mathbf{D}} \backslash F. \end{cases}$$

Then $e_n = (1 - p_F)^{1/n}$, $n \geq 1$, is a s.b.a.i. in $\mathcal{A}_F(\bar{\mathbf{D}})$. There is no difficulty in identifying $\mathcal{M}(\mathcal{A}_F(\bar{\mathbf{D}}))$ with the subalgebra $H^\infty_{\mathbf{T}\setminus F}$ consisting of all the functions in H^∞ continuously extendable to $\bar{\mathbf{D}}\setminus F$. (We follow the notation of [3].) Thus we deduce immediately from Theorem 3.2 the following result.

THEOREM 3.3. *Let Φ be a representation from $\mathcal{A}(\bar{\mathbf{D}})$ into $\mathcal{L}(E)$ and let F be a closed subset of $\mathbf{T}$ of measure 0 such that:*

$$(*_F) \qquad\qquad \Phi(\mathcal{A}_F(\bar{\mathbf{D}})) \quad \textit{is dense in } E\,.$$

Then Φ has a unique homomorphic extension to the algebra $H^\infty_{\mathbf{T}\setminus F}$. Moreover this extension is norm-continuous and of the same norm as Φ.

Remark 3.4. With respect to the applicability of Theorem 3.3 we observe that the subspace $E_0 = (\Phi(\mathcal{A}_F(\bar{\mathbf{D}}))\,E)^-$ is always invariant for T (in fact it is even a hyperinvariant subspace for T, that is, a subspace invariant for any operator commuting with T). Hence we can consider the representation $\Phi_0 : \mathcal{A}(\bar{\mathbf{D}}) \to \mathcal{L}(E_0)$ obtained by setting $\Phi_0(f) = \Phi(f)|_{E_0}$, $f \in \mathcal{A}(\bar{\mathbf{D}})$ and apply Theorem 2.3 to this representation. In other words either T has a nontrivial hyperinvariant subspace or Φ can be extended to $H^\infty_{\mathbf{T}\setminus F}$. Of course (cf. [9]) in the case of a representation Φ generated by a contraction T on a Hilbert space a much stronger result is known: either T has a nontrivial hyperinvariant subspace or Φ can be extended to H^∞.

In view of the above remark it is natural to inquire what is the maximal subalgebra of H^∞ to which Φ can be extended via Theorem 2.3. Supposing that $(*_F)$ is valid for all closed subsets F of $\mathbf{T}$ of Lebesgue measure 0 we can define (uniquely) $f(T)$ for any f in the set $\mathcal{B}_0 = \bigcup_F H^\infty_{\mathbf{T}\setminus F}$ where F runs over the collection of closed subsets of $\mathbf{T}$ of Lebesgue measure 0. It is easily checked that $\mathcal{B}_0$ is a subalgebra of H^∞ and that $f \to f(T)$ is a homomorphism. Moreover, since $\|f(T)\| \leq \|\Phi\|\,\|f\|$, $f \in \mathcal{B}_0$, Φ has a unique continuous homomorphic extension to $\mathcal{B} = \mathcal{B}_0^-$. Note that for any $f \in \mathcal{B}$ there is a sequence (F_n) of closed Lebesgue negligible subsets of $\mathbf{T}$ and a sequence (f_n) of functions such that

$f_n \in H^\infty_{\mathbf{T}\setminus F_n}$ and $\|f - f_n\| \to 0$. The set $\Gamma = \bigcap_n(\mathbf{T}\setminus F_n)$ is a G_δ-subset of $\mathbf{T}$ whose complement (in $\mathbf{T}$) is negligible and f is continuously extendable to $\mathbf{D} \cup \Gamma$, that is, in the notation of [3], $f \in H^\infty_\Gamma$. In other words $\mathcal{B} \subset \mathcal{A}$ where

$$\mathcal{A} = \cup\{H^\infty_\Gamma \,:\, \Gamma \text{ a } G_\delta\text{−subset of } \mathbf{T}\,,\; m(\mathbf{T}\setminus\Gamma) = 0\}\,.$$

It is straightforward to check that $\mathcal{A}$ itself is a closed subalgebra of H^∞. In fact, combining results and techniques of [2] and [3], the second author has proved that $\mathcal{A} = \mathcal{B}$. (This result as well as a more detailed study of $\mathcal{A}$ will be the subject of another note.)

4. EXTENSION OF REPRESENTATIONS OF UNIFORM ALGEBRAS

In this section $\mathcal{A}$ is a uniform algebra on a compact Hausdorff space X (that is, $\mathcal{A}$ is a closed unital subalgebra of $\mathbf{C}(X)$ which separates points of X), Φ is a unital representation of $\mathcal{A}$ into $\mathcal{L}(E)$, and $\mathcal{J}$ is an ideal of $\mathcal{A}$. We wish to apply Theorem 2.3 to extend Φ to $\mathcal{M}(\mathcal{J})$. First we show that $\mathcal{M}(\mathcal{J})$ itself is a uniform algebra and then state the corollary of Theorem 3.3 in this context. We conclude by developing the connection between the existence of b.a.i. and peak sets.

To avoid overly complicated notations it will be convenient to consider X as embedded in $\operatorname{Max}\mathcal{A}$, the maximal ideal space of $\mathcal{A}$, and treat elements of $\mathcal{A}$ as functions on $\operatorname{Max}\mathcal{A}$. In this setting $h(\mathcal{J})$, the hull of the ideal $\mathcal{J}$ in $\mathcal{A}$, is defined by

$$h(\mathcal{J}) = \{x \in \operatorname{Max}\mathcal{A} \,:\, f(x) = 0\,, \quad f \in \mathcal{J}\}\,.$$

Note that, by definition of $h(\mathcal{J})$, if $x \notin h(\mathcal{J})$ there exists $f \in \mathcal{J}$ such that $f(x) \neq 0$. Observe also that if g is another element of $\mathcal{J}$ such that $g(x) \neq 0$ and T is a multiplier of $\mathcal{J}$ then, from $fTg = gTf$ we get

$$Tf(x)/f(x) = Tg(x)/g(x)\,.$$

In other words for any $x \in \operatorname{Max}\mathcal{A}\setminus h(\mathcal{J})$ and any $T \in \mathcal{M}(\mathcal{J})$ we can define $T(x)$ $(\in \mathbf{C})$ such that

$$Tf(x) = T(x)\,f(x)\,, \quad x \in \operatorname{Max}\mathcal{A}\setminus h(\mathcal{J})\,, \quad f \in \mathcal{J}$$

(Of course this is the already defined value of $T(x)$ when T is a multiplier of $\mathcal{J}$ associated to an element of $\mathcal{A}$.)

Clearly $T \mapsto T(x)$ is a nontrivial multiplicative linear functional on $\mathcal{M}(\mathcal{J})$. Hence it is continuous and, moreover, $|T(x)| \leq \|T\|$. On the other hand we have

$$\begin{aligned}\|T\| &= \sup\{\|Tf\| \,:\, f \in \mathcal{J}, \|f\| \leq 1\} \\ &= \sup\{|Tf(x)| \,:\, x \in \operatorname{Max}\mathcal{A}\backslash h(\mathcal{J}), f \in \mathcal{J}, \|f\| \leq 1\} \\ &= \sup\{|T(x)|\,|f(x)| \,:\, x \in \operatorname{Max}\mathcal{A}\backslash h(\mathcal{J}), f \in \mathcal{J}, \|f\| \leq 1\} \\ &\leq \sup\{|T(x)| \,:\, x \in \operatorname{Max}\mathcal{A}\backslash h(\mathcal{J})\}\,.\end{aligned}$$

Therefore

$$\|T\| = \sup\{|T(x)| \,:\, x \in \operatorname{Max}\mathcal{A}\backslash h(\mathcal{J})\}, \quad T \in \mathcal{M}(\mathcal{J})\,.$$

This clearly shows that $\|T^2\| = \|T\|^2$ and, hence, that $\mathcal{M}(\mathcal{J})$ is a uniform algebra (cf. [6, I.5.3]). We summarize the above in the following statement.

PROPOSITION 4.1. *Let $\mathcal{J}$ be an ideal in a uniform algebra $\mathcal{A}$ on a compact Hausdorff space X. Then $\mathcal{M}(\mathcal{J})$ is a uniform algebra on some compact Hausdorff space containing* $\operatorname{Max}\mathcal{A}\backslash h(\mathcal{J})$.

In this context Theorem 2.3 takes the following form.

THEOREM 4.2. *Let Φ be a representation of the uniform algebra $\mathcal{A}$ into $\mathcal{L}(E)$ and let $\mathcal{J}$ be an ideal of $\mathcal{A}$ possessing a s.b.a.i. Suppose that*

$$(*_{\mathcal{J}}) \qquad (\Phi(\mathcal{J})\,E)^{-} = E\,.$$

Then Φ has a unique homomorphic extension to $\mathcal{M}(\mathcal{J})$. Moreover this extension is continuous with the same norm as Φ.

Remark 4.3. With respect to $(*_{\mathcal{J}})$ we observe that, just as in the case of the $\mathcal{J} = \mathcal{A}_F(\bar{\mathbf{D}})$, the subspace $E_0 = (\Phi(\mathcal{J})\,E)^{-}$ is invariant for Φ, that is, invariant for any operator in the range of Φ.

In [1] several particular cases of Theorem 4.2 are given, mostly arising from the "classical" function algebras on a compact subset X of the complex plane: $R(X)$, the closure in $C(X)$ of rational functions with poles off X, and $\mathcal{A}(X)$, the algebra of functions continuous on X and analytic on the interior of X. Besides a detailed consideration of the case where the ideal $\mathcal{J}$ consists of those functions vanishing at a point of the boundary of X (with some geometrical property "accessibility" or "nice accessibility" to ensure the existence of s.b.a.i. in $\mathcal{J}$) examples are given involving the construction of s.b.a.i. via peak sets.

Recall that a subset of P of X is called a peak set for $\mathcal{A}$ if there exists $f \in \mathcal{A}$ such that

$$\begin{cases} f|_P = 1 \\ |f(x)| < 1, & x \in X \backslash P. \end{cases}$$

As suggested by the examples in [1], there is a close connection between peak sets and b.a.i.

THEOREM 4.4. *Let $\mathcal{A}$ be a uniform algebra on a metrizable compact space X and let $\mathcal{J}$ be an ideal in $\mathcal{A}$. Then $\mathcal{J}$ has a s.b.a.i. if and only if $X \cap h(\mathcal{J})$ is a peak set for $\mathcal{A}$.*

Proof. Suppose that $X \cap h(\mathcal{J})$ is a peak set for $\mathcal{A}$. Let f be a function in $\mathcal{A}$ peaking on $X \cap h(\mathcal{J})$, that is

$$\begin{cases} f(x) = 1, & x \in X \cap h(\mathcal{J}) \\ |f(x)| < 1, & x \notin h(\mathcal{J}). \end{cases}$$

Then it is easy to verify that the sequence of functions (e_n)

$$e_n(x) = (1 - f(x))^{1/n}$$

is a s.b.a.i. for $\mathcal{J}$.

Conversely suppose that (e_n) is a s.b.a.i. for $\mathcal{J}$ and let μ be a measure on X orthogonal to $\mathcal{A}$, that is, $\int f\, d\mu = 0$, $f \in \mathcal{A}$. By [6, II.12, pp.56–57] it suffices to prove that $\int_{X \cap h(\mathcal{J})} f\, d\mu = 0$, $f \in \mathcal{A}$ to finish the proof.

For any $f \in \mathcal{A}$, we have

$$0 = \int_X f\, e_n\, d\mu = \int_{X \backslash h(\mathcal{J})} f\, e_n\, d\mu\,.$$

Note that for any $x \in X\backslash h(\mathcal{J})$ we have

$$\lim_{n\to\infty} e_n(x) = 1 \quad (\text{because of } \|fe_n - f\| \to 0)\,.$$

An application of Lebesgue's dominated convergence theorem yields

$$\int_{X\backslash h(\mathcal{J})} f\,d\mu = 0\,, \quad f \in \mathcal{A}\,,$$

and consequently

$$\int_{X\cap h(\mathcal{J})} f\,d\mu = 0$$

as desired.

Remark 4.5. If we drop the condition that X be metrizable in Theorem 4.4 it just requires some minor extra technical work to prove that $\mathcal{J}$ has a b.a.i. if and only if $X \cap h(\mathcal{J})$ is a generalized p-set for $\mathcal{A}$ in the sense of [6, II.12].

REFERENCES

[1] B. Chevreau, J. Esterle, Banach algebra techniques and spectra of functions of operators, *Operator Theory: Advances and Applications* **14** (1984), 61–79, Birkhäuser Verlag, Basel.

[2] A. Davie, T. Gamelin and J. Garnett, Distance estimates and pointwise bounded density, *Trans. Amer. Math. Soc.* **175** (1973), 37–68.

[3] J. Detraz, Algèbres de fonctions analytiques dans le disque, *Ann. Scient. Ec. Norm. Sup.* $4^{ème}$ *série*, **3** (1970), 313–352.

[4] J. Esterle, Quasimultipliers, representations of H^∞ and the closed ideal problem for commutative Banach algebras, in J.M. Bachar et al. (Ed.), *Radical Banach Algebras and Automatic Continuity*, 66–162, Lecture Notes Math. 975, Springer-Verlag, Berlin and New York, 1983.

[5] C. Foiaş, W. Mlak, The extended spectrum of completely non-unitary contractions and the spectral mapping theorem, *Studia Math.* **26** (1966), 239–245.

[6] T. Gamelin, *Uniform Algebras*, Prentice-Hall, N.J., 1965.

[7] E. Hewitt, K. Ross, *Abstract Harmonic Analysis II*, Springer-Verlag, Berlin-New York, 1970.

[8] A.M. Sinclair, Bounded approximate identities, factorization and a convolution algebra, *J. Functional Analysis*, **29** (1978), 308–318.

[9] B. Sz-Nagy, C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, Akademiai Kiado, Budapest, 1970.

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