THE NATURAL GENERALIZATION OF THE NATURAL CONDITIONS OF LADYZHENSKAYA AND URAL'TSEVA

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In the 1960's, Ladyzhenskaya and Ural'tseva [4] introduced a set of hypotheses for divergence structure elliptic operators which arise in a natural way and which lead to important estimates on the solutions of equations involving such operators. Here we discuss a more general set of hypotheses which achieve the same ends while also being the most general such, in a sense explained below.

Specifically divergence structure elliptic operators are those of the form

$$Qu = \operatorname{div} A(x,u,Du) + B(x,u,Du),$$

for some vector function A and scalar function B. Ladyzhenskaya and Ural'tseva assumed that A and B satisfied the hypotheses

(H1) $$p\cdot A(x,z,p) \geq |p|^m - a_0|z|^m - a_1,$$

(H2) $$|A(x,z,p)| \leq a_2|p|^{m-1} + a_3|z|^{m-1} + a_4,$$

(H3) $$|B(x,z,p)| \leq b_0|p|^m + b_1|p|^{m-1} + b_2|z|^{m-1} + b_3$$

for some $m > 1$ and suitable nonnegative functions a_0, a_1, a_3, a_4, b_1, b_2, b_3, and constants a_2 and b_0.

The motivation for introducing (H1)-(H3) comes from a related minimization problem: to find a function which minimizes the functional

$$I(u) = \int_\Omega F(x,u,Du)\, dx$$

over all functions u satisfying a suitable boundary condition (for example, $u = \varphi$, a known function, on $\partial\Omega$). If $F(x,z,p) = |p|^m$, then the Euler-Lagrange equation for this functional is $\mathrm{div}(|Du|^{m-2}Du) = 0$, which is elliptic if and only if $m > 1$. More is true in this case: if we differentiate out the Euler-Lagrange equation, we obtain an equation of the form $a^{ij}D_{ij}u = 0$ for some positive definite matrix (a^{ij}) with the ratio of maximum to minimum eigenvalues uniformly bounded. For short we say that the equation is uniformly elliptic. Another advantage to such structure is that the natural space for minima of the integral is the Sobolev space $W^{1,m}$ which has many useful functional analytical properties. More general structure conditions lead to the study of Orlicz spaces, which are more complicated. It was, in part, the study of more general structure conditions which led Trudinger to his investigation of Orlicz spaces [9].

Another important reason for studying such operators is that solutions of the equation $Qu = 0$ have good regularity properties.

Ladyzhenskaya and Ural'tseva showed under these hypotheses that

(1) all weak solutions are bounded if $b_0 = 0$,

(2) all bounded solutions are Hölder continuous.

If also A is differentiable with respect to the variables x,z,p and

$$\text{(H4)} \qquad A_p \geq (1+|p|)^{m-2}I, \quad |A_p||p|^2, \quad |A_z||p|, \quad |A_x| = O(|p|^m),$$

then

(3) all solutions have Hölder continuous gradient.

(Their proof was based on the corresponding results of DeGiorgi [1] for linear equations, which are included as a special case of this structure for $m = 2$.) It was later shown by Moser [5] (for linear equations), and then by Serrin [6] (for $b_0 = 0$) and by Trudinger [8] (in general) that

(4) all nonnegative solutions satisfy a Harnack-type inequality.

If we now consider the functional I with $f(x,z,p) = G(|p|)$, the Euler-Lagrange equation is $\operatorname{div}(g(|Du|)Du/|Du|) = 0$ for $g = G'$. This equation will be uniformly elliptic precisely if there are positive constants δ and g_0 such that

$$\text{(G)} \qquad \delta \leq tg'(t)/g(t) \leq g_0 \quad \text{for } t > 0.$$

Our natural conditions here are those of Ladyzhenskaya and Ural'tseva with the function t^{m-1} replaced by $g(t)$ with g as above.

That is, we assume that

(H1′) $$p \cdot A(x,z,p) \geq |p|g(|p|) - a_1|z|g(|z|) - a_2,$$

(H2′) $$|A(x,z,p)| \leq a_3 g(|p|) + a_4 g(|z|) + a_5,$$

(H3′) $$|B(x,z,p)| \leq b_0|p|g(|p|) + b_1 g(|p|) + b_2 g(|z|) + b_3.$$

(H4′) $$A_p \geq g(|p|)I/|p|,\ |A_p||p|^2,\ |A_z||p|,\ |A_x| = O(|p|g(|p|)),$$

(Such structure conditions were shown to imply a gradient bound by Simon [7].) We can show that (1), (2), (3) and (4) are true under these hypotheses also. (Of course we only need (H4)′ for (3).) Since Ladyzhenskaya and Ural′tseva use the homogeneity and multiplicative properties of power functions, their proofs must be modified to apply to these more general conditions. The necessary modifications in fact simplify the proofs of the regularity results even for the power case. Instead of the Sobolev inequality for $W^{1,m}$ functions with $m > 1$, we only need it for $m = 1$ (the case which implies the inequality for $m > 1$), and we use a weak form of Young′s inequality.

First we show that condition (G) is satisfied by nonpower functions. For example, $g(t) = t^m \ln(t + 1)$ $(m > 1)$ also satisfies condition (G) with $\delta = m$ and $g_0 = m + 1$. (In fact, $tg'(t)/g(t)$ decreases from $m + 1$ to m as t increases from 1 to ∞.) More generally there are functions g which satisfy our condition (G) with arbitrary δ and g_0 and these constants are best possible even in a neighborhood of ∞. To obtain such a function, choose numbers $\beta > \alpha > \varepsilon > 0$, define t_j inductively by

$$t_0 = 2, \ t_{j+1} = (t_j)^{(\beta+\varepsilon-\alpha)/\varepsilon},$$

and take g(t) to be t^α if $0 \le t \le t_0$, $(t_{2k})^{\alpha-\beta-\varepsilon}t^{\beta+\varepsilon}$ if $t_{2k} \le t \le t_{2k+1}$ for some nonnegative integer k, and $(t_{2k+1})^{\beta+\varepsilon-\alpha}t^{\alpha-\varepsilon}$ if $t_{2k+1} \le t \le t_{2k+2}$. It follows that condition (G) is satisfied for $\delta = \alpha - \varepsilon$ but no larger δ and $g_0 = \beta + \varepsilon$ but no smaller g_0.

To demonstrate the basic ideas involved, we prove the local boundedness of solutions under simpler hypotheses, namely B = 0 and $a_1 = a_3 = a_4 = 0$. Using ς to denote a cut-off function in a ball B_1 of radius 1, we use $\varphi = (u-k)_+\varsigma^s$ as test function in the weak form of the equation Qu = 0, and we write Σ for the subset of B_1 on which u > k. Then

$$0 = \int_\Sigma \ [\varsigma^s Du\cdot A + s(u-k)\varsigma^{s-1}D\varsigma\cdot A]$$

$$\ge \int_\Sigma \ [\varsigma^s|Du|g(|Du|) - a_2 s\varsigma^s(u-k)g(|Du|)|D\varsigma|/\varsigma].$$

Now we use Young's inequality in the form $ag(b) \le ag(a) + bg(b)$, which follows from the monotonicity of g, with $a = 2a_2s(u-k)|D\varsigma|/\varsigma$ and $b = |Du|$ to see that

$$0 \geq \int_\Sigma [\tfrac{1}{2}|Du|g(|Du|) - 2sa_2(u-k)\varsigma^{s-1}|D\varsigma|g(2a_2(u-k)|D\varsigma|/\varsigma).$$

From condition (G), we infer that $t^{-\sigma}g(t)$ is a decreasing function of t if $\sigma \geq g_0$. Hence if $s = g_0 + 1$ and K is an upper bound for $|D\varsigma|$, we have

$$\int_\Sigma \varsigma^s|Du|g(|Du|) \leq C(g_0,a_2)\int_\Sigma g(K(u-k))K(u-k).$$

Now we recall that $G' = g$ and observe that

$$|DG(K(u-k)| = K|Du|g(K(u-k))|$$

$$\leq K|Du|g(|Du|) + K[g(K(u-k))K(u-k)],$$

$$1 \leq tg(t)/G(t) \leq 1 + g_0.$$

Using these inequalities and the Sobolev inequality, we conclude that

$$\text{(E)} \qquad (\int_\Sigma \varsigma^s|G(K(u-k))|^{n/(n-1)})^{(n-1)/n} \leq CK\int_\Sigma |G(K(u-k))|.$$

Standard methods now can be used to bound u on the ball $B_{1/2}$, which is concentric to B_1 with radius 1/2. For $k > 0$ to be chosen and any positive integer N, we define

$$k_N = k(1 - 2^{-N}),\ R_N = (1 + 2^{-N})/2,\ \rho_N = 2^{-N-3},$$

and we write B(N) and A(N) for the balls concentric with B, having radii R_N and $(R_N + R_{N+1})/2$, respectively. In our estimate (E), we take $K = 1/\rho_N$ and replace k by k_N. For

$$Y_N = \int_{B(N)} G(2(u-k_N)_+)/G(k),$$

we find that $Y_{N+1} \le C2^N(Y_N)^{1+1/n}$, and therefore Y_N tends to zero as N tends to infinity if $Y_0 \le C_1$, a known constant. This inequality holds if k is chosen so that

$$\int_{B(0)} G(u_+) \le C_1$$

(note that $B(0) = B_1$), while Y_N tending to zero means that $u \le k$ on $B_{1/2}$. Therefore u is locally bounded since this argument is easily modified to handle the case that Qu = 0 only on a ball of any positive radius. The Hölder regularity and Harnack inequality follow by similar modification of the power case.

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