THE UNIQUENESS OF DIFFUSION SEMIGROUPS

Brian Jefferies

Abstract. It is shown that if the highest order co-efficients of a uniformly elliptic second order differential operator L on $\mathbb{R}^d$ are bounded and Hölder continuous, and the other coefficients are bounded and measurable, then there is at most one semigroup S acting on bounded Borel measurable functions, such that S is given by a transition function, and for all smooth functons f with compact support in $\mathbb{R}^d$, $S(t)f(x) = f(x) + \int_0^t S(s)Lf(x)\,ds$ for all $t > 0$ and $x \in \mathbb{R}^d$.

1. Introduction. One approach to the construction of a diffusion process on $\mathbb{R}^d$ generated by an elliptic second order differential operator L is to find a semigroup S of operators acting on the space of all bounded Borel measurable functions on $\mathbb{R}^d$, such that S is given by a transition function, and for all smooth functons f with compact support in $\mathbb{R}^d$,

$$S(t)f(x) = f(x) + \int_0^t S(s)Lf(x)\,ds \tag{1}$$

for all $t > 0$ and $x \in \mathbb{R}^d$. If all of the coefficients of L are Hölder continuous, then the existence and uniqueness of solutions of (1) follow from the classical theory of parabolic partial differential equations [1]. A more recent approach [5] is to construct directly, for each $x \in \mathbb{R}^d$, a probability measure P^x on the space of continuous $\mathbb{R}^d$-valued paths, so that for every smooth function f with compact support,

$$f{\circ}X_t - \int_0^t Lf{\circ}X_s\,ds, \text{ for all } t > 0$$

is a P^x-martingale with respect to the filtration $\sigma\{X_r : 0 < r \le s\}$, $s > 0$, and $P^x(g{\circ}X_0) = g(x)$ for every bounded Borel measurable function g on $\mathbb{R}^d$. For each $t \ge 0$, the random variable X_t is taken to be evaluation at time t.

A *unique* solution to the martingale problem gives a Markov process whose transition functions are associated with a semigroup satisfying (1). The existence and uniqueness of solutions to the martingale problem for strictly elliptic operators with continuous second order coefficients has been proved by Stroock and Varadhan [5], thereby establishing the existence of solutions of equation (1) for this class of operators. The *uniqueness* of solutions of equation (1) does not apparently follow from their method, which appeals directly to the sample path properties of the associated process.

To be more precise about the class of semigroups we are to deal with, the space of all bounded Borel measurable functions $\mathcal{L}^\infty(\mathbb{R}^d)$ is considered to be in duality with the space $\mathcal{M}(\mathbb{R}^d)$ of signed Borel measures on $\mathbb{R}^d$ via the pairing $< f, \nu >= \int_{\mathbb{R}^d} f\,d\nu$, for every

$f \in \mathcal{L}^\infty(\mathbb{R}^d)$ and every $\nu \in \mathcal{M}(\mathbb{R}^d)$. The collection of all Borel subsets of $\mathbb{R}^d$ is denoted by $\mathcal{B}(\mathbb{R}^d)$.

A *positive* $\mathcal{M}(\mathbb{R}^d)$*-semigroup* S is a collection $S(t)$, $t > 0$ of $\sigma(\mathcal{L}^\infty(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d))$ - continuous operators acting on $\mathcal{L}^\infty(\mathbb{R}^d)$ such that
(i) $S(t)f \geq 0$ whenever $t > 0$, $f \in \mathcal{L}^\infty(\mathbb{R}^d)$ and $f \geq 0$,
(ii) $S(t+s) = S(t)S(s)$ for all $s, t > 0$, and
(iii) there exists $\lambda_S \geq 0$ such that for all $\lambda > \lambda_S$ and $f \in \mathcal{L}^\infty(\mathbb{R}^d)$, the map $t \mapsto e^{-\lambda t} < S(t), \nu >$, $t > 0$ is integrable on $(0, \infty)$ for every $\nu \in \mathcal{M}(\mathbb{R}^d)$, and there exists $R(\lambda)f \in L^\infty(\mathbb{R}^d)$ such that $< R(\lambda)f, \nu >= \int_0^\infty e^{-\lambda t} < S(t)f, \nu > dt$ for all $\nu \in \mathcal{M}(\mathbb{R}^d)$.

The family of operators $R(\lambda) : \mathcal{L}^\infty(\mathbb{R}^d) \mapsto \mathcal{L}^\infty(\mathbb{R}^d)$, $\lambda > \omega_S$ is called the *resolvent* of S. By monotone convergence, for each $\lambda > \omega_S$, $R(\lambda)$ is $\sigma(\mathcal{L}^\infty(\mathbb{R}^d), M(\mathbb{R}^d))$-continuous [**2**, Proposition 1]. The positivity condition is a lower bound, and the existence of a resolvent is a condition on the growth of the semigroup at 0 and ∞.

If S is a positive $\mathcal{M}(\mathbb{R}^d)$-semigroup, for each $t > 0$ define the function $p_t : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \mapsto [0, \infty)$ by $p_t(x, A) = S(t)\chi_A(x)$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$. It follows that $A \mapsto p_t(x, A)$, $A \in \mathcal{B}(\mathbb{R}^d)$ is σ-additive for each $x \in \mathbb{R}^d$, $x \mapsto p_t(x, A)$, $x \in \mathbb{R}^d$ is Borel measurable for each $A \in \mathcal{B}(\mathbb{R}^d)$ and $p_{t+s}(x, A) = \int_{\mathbb{R}^d} p_t(y, A) p_s(x, dy)$ for all $s, t > 0$, $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$. Furthermore, for each $\lambda > \omega_S$ the function $x \mapsto \int_0^\infty e^{-\lambda t} p_t(x, A)\, dt$, $x \in \mathbb{R}^d$ exists, and is Borel measurable.

Conversely, given a collection of functions $p_t : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \mapsto [0, \infty)$, $t > 0$ such that $(x, t) \mapsto p_t(x, A)$ is jointly measurable for each $A \in \mathcal{B}(\mathbb{R}^d)$, and if p_t, $t > 0$ has the appropriate properties, a positive $\mathcal{M}(\mathbb{R}^d)$-semigroup can be constructed in a similar fashion.

Now we ignore equation (1) and work exclusively with the resolvent operators $R(\lambda)$, $\lambda > \omega_S$, because equation (1) is equivalent to the requirement that for *some* $\lambda > \omega_S$, $R(\lambda)(\lambda - L)f = f$ for all smooth functions f with compact support [**2**, Theorem 1].

In section 2, it is shown that in the case that $R(\lambda)$ exists, when it is viewed as an $\mathcal{L}^\infty(\mathbb{R}^d)$-valued vector measure it is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$, provided that the differential operator L has measurable coefficients and is strictly elliptic—the highest order coefficients of L are stricly positive definite at each point of $\mathbb{R}^d$. No continuity assumptions on the coefficients of L are necessary.

Further restrictions are imposed on the second order elliptic differential operator L in section 3 to ensure that (1) has at most one solution. If the coefficients of L are bounded and measurable, if it is uniformly elliptic, and if the second order coefficients are Hölder continuous, then the conditions are satisfied. Of course, these conditions also ensure the existence of a solution of (1) (see [**5**], or [**2**],[**4**] for a direct proof).

Uniqueness can be obtained with only the uniform continuity of the second order coefficients if the domain of L is taken to be larger than the natural domain $C_\kappa^\infty(\mathbb{R}^d)$ of smooth functions of compact support [**2**].

2. Absolute continuity. Suppose that $R(\lambda)$, $\lambda > \omega_S$ is the resolvent of a positive $\mathcal{M}(\mathbb{R}^d)$-semigroup, and for some $\lambda > \omega_S$, $R(\lambda)(\lambda - L)f = f$ for all $f \in C_\kappa^\infty(\mathbb{R}^d)$. This section is devoted to showing that if K is any compact subset of $\mathbb{R}^d$ of Lebesgue measure zero, then there exists an arbitrarily small function $f \in C_\kappa^\infty(\mathbb{R}^d)$ such that the sup-norms of the partial derivatives of f are bounded by one, and $(\lambda - L)f$ is arbitrarily large on K. An appeal to monotone convergence then shows that the vector measure $A \mapsto R(\lambda)\chi_A$, $A \in \mathcal{B}(\mathbb{R}^d)$ is absolutely continuous with respect to Lebesgue measure. This is the only point at which the positivity of $R(\lambda)$ is used. Some notation used in the present section follows.

Let $n = 1, 2, \ldots$. For a vector $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, the Euclidean norm of x is denoted by $|x| = (\sum_{i=1}^n x_i{}^2))^{1/2}$. The distance between two sets $A, B \subseteq \mathbb{R}^n$, is defined by $d(A, B) = \inf\{|x - y| : x \in A, y \in B\}$. Given two measurable functions f , g on $\mathbb{R}^n$, the function $f * g$, when it exists, is defined by $f * g(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\, dy$, $x \in \mathbb{R}^n$. The Lebesgue measure of a Borel set $A \in \mathbb{R}^n$ is denoted by $|A|$.

LEMMA 1. *Let K be a compact subset of $\mathbb{R}^d$. For every open set U containing K, there exists a non-negative smooth function f with compact support such that $f(x) > 0$ and $f'(x) \neq 0$ for all $x \in K$, and* supp$f \subset U$.

PROOF: Because K is compact, there exist bounded open sets V_i, W_i, $i = 1, \ldots, n$ contained in U such that for each $i = 1, \ldots, n$, $\overline{V_i} \subseteq W_i$, and V_i, $i = 1, \ldots, n$ covers K. Let $\epsilon < \min\{d(\overline{V_i}, W_i{}^c) : i = 1, \ldots, n\}$. Let ϕ be a non-negative, smooth function with support contained in the unit ball of $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} \phi(x)\, dx = 1$, and define $\phi_\epsilon(x) = \frac{1}{\epsilon^d}\phi(\frac{x}{\epsilon})$. For each $i = 1, \ldots, n$, suppose that $f_i : \mathbb{R}^d \mapsto \mathbb{R}$ is defined by $f_i(x) = \sum_{j=1}^d a_{i,j}x_j + b_i$ for every $x \in V_i$, and $f_i(x) = 0$ for every $x \notin V_i$. The numbers $a_{i,j} > 0$, b_i, $i, j = 1, \ldots, d$ are chosen so that $b_i > -inf\{\sum_{j=1}^d a_{i,j}x_j : x \in W_i\}$. The function $\sum_{i=1}^n \phi_\epsilon * f_i$ has the required properties.

LEMMA 2. *Let K be a compact subset of $(0, \infty)$ of zero Lebesgue measure . For every $\epsilon > 0$, there exists a smooth function f on $\mathbb{R}$ such that* supp$f \subseteq [0, \infty)$, *$0 < f(x) \leq \epsilon$ for all $x \in (0, \infty)$, and $f'(x) \geq \frac{1}{\epsilon}$ for all $x \in K$.*

PROOF: Let $\delta = \frac{\inf K}{2}$. The set K is compact and of zero Lebesgue measure, so there exist bounded open subsets U_i , V_i, $i = 1, \ldots, n$ of (δ, ∞) such that $K \subseteq \bigcup_{i=1}^n U_i$, $\sum_{i=1}^n |V_i| < \epsilon^2$, and $\overline{U_i} \subseteq V_i$ for all $i = 1, \ldots, n$. Let $\gamma < \min\{\delta, d(\overline{U_i}, V_i{}^c) : i = 1, \ldots, n\}$. Let ϕ be a non-negative, smooth function with support contained in the interval $(0, 1)$ of $\mathbb{R}$ such that $\int_{\mathbb{R}} \phi(x)\, dx = 1$, and define $\phi_\gamma(x) = \frac{1}{\gamma}\phi(\frac{x}{\gamma})$. Suppose that g is a smooth function positive and less than 1 on $(0, \delta)$, and supported by $[0, \delta]$. Let

$$f(x) = \frac{1}{2\epsilon}\sum_{i=1}^n \int_0^x \phi_\gamma * \chi_{V_i}(t)\, dt + \frac{\epsilon}{2}g(x)$$

for all $x \geq 0$, and let $f(x) = 0$ for all $x < 0$. Then for all $x \in \mathbb{R}$,

$$|f(x)| \leq \frac{1}{2\epsilon} \sum_{i=1}^{n} \int_{\mathbb{R}} \chi_{V_i}(t)\, dt + \frac{\epsilon}{2} \leq \epsilon.$$

Because $\gamma < \delta$, and for each $i = 1, \ldots, n$ the set V_i, is contained in (δ, ∞), $\phi_\gamma * \chi_{V_i}$ vanishes on $(-\infty, 0]$, so f is smooth. Moreover

$$f'(x) = \frac{1}{\epsilon} \sum_{i=1}^{n} \phi_\gamma * \chi_{V_i}(x) + \frac{\epsilon}{2} g'(x).$$

If $x \in U_i$, then $\phi_\gamma * \chi_{V_i}(x) = 1$, so surely $f'(x) \geq \frac{1}{\epsilon}$. Therefore, for every $x \in K, f'(x) \geq \frac{1}{\epsilon}$.

For a smooth function f with compact support in $\mathbb{R}^d$, let $\|f\|_{2,\infty}$ be the sum of the sup-norms of f and its partial derivatives of order less than or equal to two.

LEMMA 3. *Let K be a compact subset of $\mathbb{R}^d$ of zero Lebesgue measure. For every open set U containing K, and every $\epsilon > 0$, there exists a smooth non-negative function f with compact support, such that supp$f \subseteq U$, $|f(x)| \leq \epsilon$, $|f'(x)| \leq 1$ for all $x \in \mathbb{R}^d$, and for every $x \in K$, $\lambda > 0$, $\Lambda > 0$,*

$$\sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq \frac{1}{\epsilon}\lambda - \Lambda$$

for every matrix $[a_{i,j}]_{i,j=1}^d$ such that $\lambda|\xi|^2 \leq \sum_{i,j=1}^d a_{i,j}\xi_i\xi_j \leq \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^d$.

PROOF: There exists a number $C \geq 1$ depending only on the dimension d such that for every matrix $[a_{i,j}]_{i,j=1}^d$ and every number $\Lambda > 0$ with $|\sum_{i,j=1}^d a_{i,j}\xi_i\xi_j| \leq \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^d$, $\sum_{i,j=1}^d |a_{i,j}| \leq C\Lambda$. Choose g from Lemma 1. Then $g(K)$ is a compact subset of $(0, \infty)$ of Lebesgue measure zero, and $g' \neq 0$ on K. Let $a = \inf\{|g'(x)|^2 : x \in K\}$. Set

$$\gamma = \min\{\epsilon a, \frac{1}{C\|g\|_{2,\infty}}, 1\}.$$

By Lemma 2, there exists a smooth function h with compact support such that supp $h \subseteq [0, \infty)$, $0 < h(x) \leq \gamma$ for all $x > 0$, and $h'(x) \geq \frac{1}{\gamma}$ for all $x \in g(K)$. Let ρ be a smooth function with compact support contained in $(0, \infty)$ such that $0 \leq \rho \leq 1$, ρ is equal to 1 in a neighbourhood of $g(K)$, and $|\text{supp } \rho| \leq \epsilon$. Set

$$u(x) = \int_0^x \rho(t)h(t)\, dt$$

for every $x \geq 0$, and $u(x) = 0$ elsewhere. Define $f = u \circ g$. Firstly, $|f(x)| \leq \int_0^\infty \rho(t)\, dt \leq \epsilon$, for all $x \in \mathbb{R}^d$. Now $f' = u' \circ g g' = (\rho h) \circ g g'$, so $|f'(x)| \leq \gamma |g'(x)| \leq 1$ for all $x \in \mathbb{R}^d$. For each $x \in \mathbb{R}^d$,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = (\rho' h + \rho h')(g(x)) \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j}(x) + \rho h(g(x)) \frac{\partial^2 g}{\partial x_i \partial x_j}(x).$$

For $x \in K$, $\rho'(g(x)) = 0$ because ρ is constant in a neighbourhood of K. Furthermore, by virtue of the choice of the number γ, $|\rho h(g(x)) \frac{\partial^2 g}{\partial x_i \partial x_j}(x)| \leq \frac{1}{C}$. Therefore for all $x \in K$,

$$\begin{aligned} \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) &\geq h'(g(x)) \sum_{i,j=1}^{d} a_{i,j} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j}(x) - \frac{1}{C} \sum_{i,j=1}^{d} |a_{i,j}| \\ &\geq \frac{1}{\gamma} \sum_{i,j=1}^{d} a_{i,j} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j}(x) - \Lambda \geq \frac{1}{\epsilon}\lambda - \Lambda \end{aligned}$$

Let $a_{i,j}$, $i,j = 1,\ldots,d$, b_i, $i = 1,\ldots,d$ and c be Borel measurable functions on $\mathbb{R}^d$. For each $x \in \mathbb{R}^d$, the matrix $[a_{i,j}(x)]_{i,j=1}^d$ is symmetric and $\sum_{i,j=1}^d a_{i,j}(x)\xi_i\xi_j \geq 0$ for all $\xi \in \mathbb{R}^d$. The operator L acting on all smooth functions with compact support in $\mathbb{R}^d$, and with values in the Borel measurable functions is defined by

$$L = \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c$$

Let $\mathcal{T}_+$ denote the collection of all functions $p : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \mapsto [0,\infty)$ such that for each $x \in \mathbb{R}^d$, $A \mapsto p(x,A)$, $A \in \mathcal{B}(\mathbb{R}^d)$ is σ-additive, and for each $A \in \mathcal{B}(\mathbb{R}^d)$, the function $x \mapsto p(x,A)$, $x \in \mathbb{R}^d$ is Borel measurable. For each $p \in \mathcal{T}_+$, we can define an operator $T_p : \mathcal{L}^\infty(\mathbb{R}^d) \mapsto \mathcal{L}^\infty(\mathbb{R}^d)$ by setting for each $f \in \mathcal{L}^\infty(\mathbb{R}^d)$, $T_p f(x) = \int_{\mathbb{R}^d} f(y)\, p(x,dy)$, for every $x \in \mathbb{R}^d$.

The operator T_p may be extended to a larger space of T_p-integrable functions in the sense of vector measures, as follows. A function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is called *T_p-integrable* if for every $x \in \mathbb{R}^d$, f is $p(x,\cdot)$-integrable, and the function $x \mapsto \int_{\mathbb{R}^d} f(y)\, p(x,dy)$, $x \in \mathbb{R}^d$ is bounded and Borel measurable.

Given a Borel measure ν on $\mathbb{R}^d$, T_p is said to be *absolutely continuous* with respect to ν (written $T_p \ll \nu$) if for every Borel set A such that $\nu(A) = 0$, $T_p \chi_A = 0$. Let μ be the Lebesgue measure on $\mathbb{R}^d$. If $R(\lambda)$, $\lambda > \omega_S$ is the resolvent of a positive $\mathcal{M}(\mathbb{R}^d)$-semigroup, then as indicated above, for each $\lambda > \omega_S$, there exists $p \in \mathcal{T}_+$ such that $R(\lambda) = T_p$.

PROPOSITION 1. *Let $\lambda > 0$. Suppose that $p \in \mathcal{T}_+$, and for every smooth function f with compact support in $\mathbb{R}^d$, the Borel measurable function $(\lambda - L)f$ is T_p-integrable, and $T_p(\lambda - L)f = f$. Then $T_p \ll \mu$.*

PROOF: It is sufficient to prove that for any compact subset K of $\mathbb{R}^d$ with zero μ-measure, $T_p\chi_K = 0$, because any Borel measure on $\mathbb{R}^d$ is compact inner-regular. Let U be a bounded open set containing K. According to Lemma 3, for each $n = 1, 2, \dots$ there exists a smooth non-negative function f_n with compact support contained in U, such that $|f_n(x)| \le \frac{1}{n}$, $|f_n'(x)| \le 1$ for all $x \in \mathbb{R}^d$, and for every $x \in K$, $\lambda > 0$ and $\Lambda > 0$,

$$\sum_{i,j=1}^{d} A_{i,j} \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x) \ge n\lambda - \Lambda$$

for every matrix $[A_{i,j}]_{i,j=1}^d$ such that

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{d} A_{i,j}\xi_i\xi_j \le \Lambda|\xi|^2$$

for all $\xi \in \mathbb{R}^d$.

Let ϕ be a smooth function of compact support such that $\phi \ge f_n$ for each $n = 1, 2, \dots$. Then $T_p(\lambda - L)(\phi - f_n) = (\phi - f_n)$ for each $n = 1, 2, \dots$, and $\lim_{n\to\infty} T_p(\lambda - L)(\phi - f_n)(x) = \phi(x)$, uniformly for all $x \in \mathbb{R}^d$.

By monotone convergence, for each $x \in \mathbb{R}^d$ the function $\liminf_{n\to\infty}(\lambda - L)(\phi - f_n)$ is $p(x, \cdot)$-integrable, so it is finite $p(x, \cdot)$-a.e.. For every $y \in \mathbb{R}^d$ the matrix $[a_{i,j}(y)]_{i,j=1}^d$ is positive-definite, so $\liminf_{n\to\infty}(\lambda - L)(\phi - f_n)(y) = \infty$ for all $y \in K$. Consequently, $p(x, K) = 0$ for each $x \in \mathbb{R}^d$, so that $T_p\chi_K = 0$.

3. Uniqueness of the resolvent operators. We now suppose that $a_{i,j}$, $i, j = 1, \dots, d$, b_i, $i = 1, \dots, d$, c are bounded Borel measurable functions on $\mathbb{R}^d$, where for each $x \in \mathbb{R}^d$, the matrix $[a_{i,j}(x)]_{i,j=1}^d$ is symmetric, there exists $\lambda > 0$ such that $\sum_{i,j=1}^d a_{i,j}(x)\xi_i\xi_j \ge \lambda|\xi|^2$ for all $x, \xi \in \mathbb{R}^d$, and for each $i, j = 1, \dots, d$, the function $a_{i,j}$ is uniformly continuous. For each $r > 0$, set

$$\omega(r) = \sup\{|a_{i,j}(x) - a_{i,j}(y)| : i, j = 1, \dots, d, x, y \in \mathbb{R}^d, |x - y| \le r\}.$$

The space of all equivalence classes of bounded Borel measurable functions on $\mathbb{R}^d$ is denoted by $L^\infty(\mathbb{R}^d)$.

THEOREM 1. *If there exists $1 < q < \infty$ such that*

$$\sup_{0 < s \le 1} \frac{\left(\int_0^s \omega(r)^q r^{d-1}\, dr\right)^{1/q}}{s^d} < \infty \tag{2}$$

then there exists $\Lambda > 0$ such that $(\lambda - L)C_\kappa^\infty(\mathbb{R}^d)$ is weak-dense in $L^\infty(\mathbb{R}^d)$ for all $\lambda > \Lambda$.*

PROOF: Let $1 \leq r < \infty$. Denote by $W^{2,r}(\mathbb{R}^d)$ the space of (equivalence classes of) functions f on $\mathbb{R}^d$, such that f and its distributional derivatives of order less than or equal to two are r-integrable functions on $\mathbb{R}^d$. The sum of the r-norms of f and these derivatives defines the norm of $W^{2,r}(\mathbb{R}^d)$. Let $L^r(\mathbb{R}^d)$ be the space of r-integrable functions on $\mathbb{R}^d$. The sup-norm on bounded functions defined on $\mathbb{R}^d$ is denoted by $\|\cdot\|_\infty$.

Under the assumption that the highest order co-efficients of the uniformly elliptic operator L are uniformly continuous, it follows that for all $1 < r < \infty$, there exists Λ_r such that for each $\lambda > \Lambda_r$, $\lambda - L$ is the restriction to $C_\kappa^\infty(\mathbb{R}^d)$ of a bijective operator $\lambda - \overline{L_r} : W^{2,r}(\mathbb{R}^d) \mapsto L^r(\mathbb{R}^d)$ [2].

Because $L^u((0,s))$ embeds in $L^v((0,s))$ for all $1 \leq v \leq u \leq \infty$ and $s > 0$, we can choose $q > 1$ so small that condition (2) is satisfied, and if $1 < r < \infty$ is defined by $\frac{1}{r} + \frac{1}{q} = 1$, then $r > \max(d, 1)$. Now fix $\lambda > \Lambda_r$. Let g be a continuous function with compact support in $\mathbb{R}^d$ and set $f = (\lambda - \overline{L}_r)^{-1} g$.

The partial derivatives $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, d$ of f belong to $L^r(\mathbb{R}^d)$. By the Sobolev embedding theorem, $\dfrac{\partial f}{\partial x_i}$, $i = 1, \ldots, d$ and f have representatives which are continuous functions vanishing at infinity [3 p. 124].

Let ρ be a non-negative smooth function with support in the unit ball of $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} \rho(x)dx) = 1$. Define $\rho_n(x) = n^d \rho(nx)$ for all $n = 1, 2, \ldots$ and $x \in \mathbb{R}^d$. It follows that there exists a number $M > 0$ such that $\|\rho_n * (b_i \frac{\partial f}{\partial x_i})\|_\infty \leq M$, $\|b_i \rho_n * \frac{\partial f}{\partial x_i}\|_\infty \leq M$, $\|\rho_n * (cf)\|_\infty \leq M$, $\|c\rho_n * f\|_\infty \leq M$, for every $i = 1, \ldots, d$, $n = 1, 2, \ldots$. Moreover, for any bounded measurable function ϕ on $\mathbb{R}^d$, $\rho_n * \phi$ converges to ϕ almost everywhere as $n \to \infty$ [3, p. 63]. Now $(\lambda - L)\rho_n * f = \rho_n * g + (\lambda - L)\rho_n * f - \rho_n * (\lambda - \overline{L_r})f$, and $(\lambda - L)\rho_n * f$ belongs to the weak*-closure of $(\lambda - L)C_\kappa^\infty(\mathbb{R}^d)$ in $L^\infty(\mathbb{R}^d)$ for each $n = 1, 2, \ldots$.

Because $\rho_n * g$ converges to g uniformly on $\mathbb{R}^d$ as $n \to \infty$, by dominated convergence it is sufficient to show that there exists $K > 0$ such that $\|\rho_n * (\lambda - \overline{L_r})f - (\lambda - \overline{L_r})\rho_n * f\|_\infty \leq K$ for all $n = 1, 2, \ldots$, and $\rho_n * (\lambda - \overline{L_r})f - (\lambda - \overline{L_r})\rho_n * f \to 0$ almost everywhere as $n \to \infty$. The weak*-closure of $(\lambda - L)C_k^\infty(\mathbb{R}^d)$ in $L^\infty(\mathbb{R}^d)$ then contains the collection of all continuous functions of compact support on $\mathbb{R}^d$, which we know to be weak*-dense in $L^\infty(\mathbb{R}^d)$.

For each $x \in \mathbb{R}^d$,

$$\begin{aligned}
\rho_n * (\lambda - \overline{L_r})f(x) - (\lambda - \overline{L_r})\rho_n * f(x) = \int_{\mathbb{R}^d} \rho_n(x-y)[a_{i,j}(x) - a_{i,j}(y)] \frac{\partial^2 f}{\partial x_i \partial x_j}(y)\,dy \\
- O_n(x)
\end{aligned}$$

$$\begin{aligned}
O_n(x) = \sum_{i=1}^{d} \rho_n * (b_i \frac{\partial f}{\partial x_i})(x) - b_i \rho_n * \frac{\partial f}{\partial x_i}(x) \\
+ \rho_n * (cf)(x) - c\rho_n * f(x)
\end{aligned}$$

where $O_n \to 0$ almost everywhere as $n \to \infty$, and $\sup_{n\in\mathbb{N}} \|O_n\|_\infty < \infty$. Estimating the first term by Hölder' s inequality,

$$|\int_{\mathbb{R}^d} \rho_n(x-y)[a_{i,j}(y) - a_{i,j}(x)]\frac{\partial^2 f}{\partial x_i \partial x_j}(y)\,dy| \le \int_{\mathbb{R}^d} \rho_n(x-y)\omega(x-y)|\frac{\partial^2 f}{\partial x_i \partial x_j}(y)|\,dy$$
$$\le \|\rho_n\omega\|_q \|\frac{\partial^2 f}{\partial x_i \partial x_j}\|_r.$$

There exists $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \|\rho_n\omega\|_q \le C\|\rho\|_\infty \sup_{n\in\mathbb{N}} n^d (\int_0^{\frac{1}{n}} \omega(r)^q r^{d-1}\,dr)^{1/q},$$

which is finite by condition (2). Moreover, for every $\phi \in L^r(\mathbb{R}^d)$, $\rho_n * \phi \to \phi$ in $L^r(\mathbb{R}^d)$, and almost everywhere on $\mathbb{R}^d$ as $n \to \infty$ [3, p. 63]. It follows that

$$\sup_{n\in\mathbb{N}} \|\rho_n * (\lambda - \overline{L_r})f - (\lambda - \overline{L_r})\rho_n * f\|_\infty < \infty,$$

and $\rho_n * (\lambda - \overline{L_r})f - (\lambda - \overline{L_r})\rho_n * f \to 0$ almost everywhere as $n \to \infty$.

COROLLARY 1. *If there exists $1 < q < \infty$ such that*

$$\sup_{0<s\le 1} \frac{(\int_0^s \omega(r)^q r^{d-1}\,dr)^{1/q}}{s^d} < \infty,$$

then there exists $\Lambda > 0$ such that for all $\lambda > \Lambda$, there is at most one $p \in T_+$, such that for every smooth function f with compact support in $\mathbb{R}^d$, $T_p(\lambda - L)f = f$.

PROOF: By Proposition 1 and the Radon-Nikodým theorem, for each $x \in \mathbb{R}^d$, $p(x,\cdot)$ has a density g_x with respect to the Lebesgue measure μ on $\mathbb{R}^d$. Suppose that $\widetilde{p} \in T_+$, and for every smooth function f with compact support in $\mathbb{R}^d$, $T_{\widetilde{p}}(\lambda - L)f = f$. Denote the density of $\widetilde{p}(x,\cdot)$ with respect to μ by h_x for each $x \in \mathbb{R}^d$.

Then $\int_{\mathbb{R}^d}(\lambda - L)f(y)(g_x(y) - h_x(y))\,dy = 0$ for all smooth functions f of compact support. Because $(\lambda - L)C_k^\infty(\mathbb{R}^d)$ is weak*-dense in $L^\infty(\mathbb{R}^d)$, $g_x = h_x$ almost everywhere, so that $p = \widetilde{p}$.

COROLLARY 2. *Suppose that there exists $C > 0$, $\alpha > 0$ such that for each $i, j = 1, \ldots, d$, $|a_{i,j}(x) - a_{i,j}(y)| \le C|x-y|^\alpha$ for all $x, y \in \mathbb{R}^d$. Then there exists $\Lambda > 0$ such that for all $\lambda > \Lambda$, there is at most one $p \in T_+$, such that for every smooth function f with compact support in $\mathbb{R}^d$, $T_p(\lambda - L)f = f$.*

PROOF: If $\alpha \ge d$, then condition (2) is satisfied for any $q > 1$, otherwise take $q = \frac{d}{d-a}$.

If $\omega(r) = -\frac{1}{\ln r}$ for all $0 < r < 1$, then (2) is not true.

Problem. For (1) to have a unique solution, is it sufficient that L be strictly elliptic with bounded measurable coefficients, and continuous second order coefficients?

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Department of Mathematics
The University of Wollongong
P.O. Box 1144 Wollongong
N.S.W. 2500
AUSTRALIA

Present Address: **School of M.P.C.E.**
Macquarie University
N.S.W. 2109
AUSTRALIA

Future Address: **School of Mathematics**
The University of N.S.W.
P.O. Box 1, Kensington
N.S.W. 2033
AUSTRALIA

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