MIXED VOLUMES AND CONNECTED VARIATIONAL PROBLEMS

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ABSTRACT. The recent achievements concerning m-curvature equations gave a new point of view to the geometrical theory of mixed volumes of convex bodies which was developed by A.D. Aleksandrov in 1938-1940. A principal goal of this paper is to pose corresponding variational problems correctly and to formulate sufficient conditions for the existence of minimizers.

1. MIXED VOLUMES. Let X, P be two n-dimensional Euclidean spaces and $u(x)$, $v(p)$ be a pair of functions from C^2. Define mappings H_u, H_v as follows:

$$H_u : X \to P\,, \quad p = u_x\,,$$

$$H_v : P \to X\,, \quad x = v_p\,.$$

We have a composition $H_{vu} = H_v \circ H_u$ and

$$H_{vu} : X \to \tilde{X} \subset X\,.$$

The mapping H_{vu} generates an exterior n-form $\tilde{\omega}_n = d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n$. We mix this one with $\omega_n = dx^1 \wedge \ldots \wedge dx^n$ as was done in [1]

$$\omega_{m,n-m}[v;u] = \frac{1}{\binom{n}{m}} \sum \sigma(i)\, d\tilde{x}^{i_1} \wedge \ldots \wedge d\tilde{x}^{i_m} \wedge dx^{i_{m+1}} \wedge \ldots \wedge dx^{i_n} \tag{1}$$

where $\sigma(i)$ is 1 or -1 in accordance with the transposition $i = (i_1 \ldots i_n)$ being even or odd and $i_1 < \ldots < i_m$, $i_{m+1} < \ldots < i_n$.

The exterior n-form (1) may be written in a more compact form as

$$\omega_{m,n-m}[v;u] = \mu_m^v[u]\,\omega_n\,. \tag{2}$$

* The support of the Centre for Mathematical Analysis (Canberra).

The operator $\mu_m^v[u]$ defined by (2) is a differential operator of the second order generated by u, v. As $\omega_{m,n-m}$ is a volume in some sense, it is natural to call $\mu_m^v[u]$ an operator density and to consider such u, v that $\mu_m^v[u] \geq 0$.

Examples. (i) $v = \frac{1}{2}\, p^2$, $u(x)$ is any function from C^2. Then

$$\mu_m^v[u] \equiv \mu_m^D[u] = \frac{1}{\binom{n}{m}}\, [u_{xx}]_m$$

where $[u_{xx}]_m$ is a sum of all m-order minors of the Hessian matrix u_{xx}. The set of non-negativity of μ_m^D contains a cone

$$K_m^D = \{u \in C^2;\ [u_{xx}]_i > 0,\ \ i = 1, \dots, m\} \tag{3}$$

as a connected component (2). In the case $m = n$ K_n^D coincides with the set of convex functions.

(ii) $v = \sqrt{1+p^2}$, $u(x)$ is any function from C^2. Then

$$\mu_m^v[u] \equiv \mu_m[u] = \frac{1}{\binom{n}{m}}\, S_m(k)$$

where $S_m(k)$ is an elementary symmetric function of the principal curvatures $k = (k^1 \dots k^n)$ of the graph $(x, u(x))$. A cone similar to (3) is

$$K_m = \{u \in C^2;\ S_i(i) > 0,\ \ i = 1, \dots, m\} \tag{4}$$

in this case [2]. As $\mu_m[u]$ has a simple geometrical sense, we shall use sometimes the notation $\mu_m[\Gamma]$ or $\mu_m[\partial\Omega]$ where Γ, $\partial\Omega$ are surfaces.

(iii) If $v(p)$ is a Lagrange transformation of the convex function $u(x)$

$$v(p) = x^i p^i - u\,, \quad x = v_p\,,$$

then $\mu_m^v[u] = u$, [1].

2. VARIATIONAL PROBLEMS. As soon as $\mu_m^v[u]$ was interpreted as the density of a measure it was reasonable to consider a family of functionals

$$\mathcal{I}_m^v(u) = \int v(u_x)\, \omega_{m,n-m}^v \equiv \int_\Omega v(u_x)\, \mu_m^v[u]\, dx\,.$$

Paper [1] contains the following assertion.

Theorem 1. *An equality*

$$\frac{\delta \mathcal{I}_m^v}{\delta u} = -(n-m)\ \mu_{m+1}^v[u]$$

holds for any $u, v \in C^2$.

Therefore an equation

$$\mu_{m+1}^v[u] = H_m(x, u) \tag{5}$$

would be the Euler-Lagrange equation for the functional

$$I_m^v[u] = \mathcal{I}_m^v(u) + (n-m) \int_\Omega f(x,u)\,dx$$

if $H_m = \partial f / \partial u$. As a particular case we get the $(m+1)$-curvature equation corresponding to the generalized area functional

$$I_m(u) = \int_\Omega \left(\sqrt{1+u_x^2}\ \mu_m[u] + (n-m)\,f(x,u)\right) dx\,. \tag{6}$$

However the integrands of these functionals depend on the second derivatives of $u(x)$ and we cannot hope to proceed in the usual way with them. The first obstacle is a mixed type of equation (5) in $C^2(\Omega)$. But it would be an elliptic type in the cone $K_m^v = \{u \in C^2(\bar{\Omega})\,; \mu_i^v[u] > 0\,, i = 1, \ldots, m\}$ if $v(p)$ is convex [2]. The second obstacle is boundary conditions. To make the situation clear we shall take as an example the functional (6) and its Euler-Lagrange equation

$$\mu_{m+1}[u] = H_{m+1}(x,u)\,. \tag{7}$$

Connect with any $u \in C^2$ a set

$$\mathcal{M}_u = \left\{v \in C^2(\bar{\Omega})\,; v\big|_{\partial\Omega} = \varphi(x)\,, v_n - u_n\big|_{\partial\Omega} \le 0\right\}$$

where $\varphi \in C^2(\partial\Omega)$ is a known function, v_n means a derivative along the inner normal to $\partial\Omega$ here and further. $\mathcal{M}_u$ contains $u(x)$ if $u\big|_{\partial\Omega} = \varphi(x)$. The following proposition has been proved in [3].

Lemma 2. *Let $u \in C^2(\bar{\Omega})$ be a minimizer for $I_m(u)$ on the set $\mathcal{M}_u$. Assume that $\partial\Omega \in C^2$ is a closed surface in R^n, $\varphi \in C^2(\partial\Omega)$, $f \in C^1(\Omega \times R^1)$. Then $u(x)$ is a solution of the equation (7) and the inequality*

$$\frac{\partial \mu_m[u]}{\partial u_{nn}} \geq 0\,, \quad x \in \partial\Omega \tag{8}$$

is fulfilled.

The value $\partial\mu_m[u]/\partial u_{nn}$ depends on the derivatives of φ, $\partial\Omega$ and u_n only. Since we may look at (8) as the additional assumption on u_n, it is reasonable to consider (8) as the second boundary condition in some sense. But it is not conditional on $u(x)$ in the end of ends. For example if $\varphi(x) = 0$, then (8) is equivalent to

$$(-u_n)^{m-1}\,\mu_{m-1}[\partial\Omega] \geq 0 \tag{9}$$

as it was shown in [4]. Since we have $u_n \leq 0$ on $\partial\Omega$ for any $u \in K_{m+1}$, $u\big|_{\partial\Omega} = 0$, inequality (9) becomes a condition on the type of boundary.

Lemma 3 contains sufficient conditions for μ to be a minimizer [3].

Lemma 3. *Let $u \in K_{m+1}$ be a solution of the equation (7) and $u\big|_{\partial\Omega} = \varphi(x)$. Assume that Ω is a bounded domain, $\partial\Omega$, $\varphi \in C^2$, $H_m \in C^1(\bar{\Omega} \times R^1)$, $\partial H_m/\partial u \geq 0$. Then $u(x)$ gives a strict local minimum to I_m with $f = -\int_u^{\varphi_1} H_m(x,t)\,dt$, $\varphi_1 =$ $\max_{\partial\Omega} \varphi$, on the set $\mathcal{M}_u$.*

We see that the principal question in this subject is the solvability of the corresponding Dirichlet problem.

3. EXISTENCE THEOREM. The recent achievements in the theory of m-curvature equations [3]–[9] lead to the following assertion.

Theorem 4. *Let $1 \leq m \leq n-1$, $\ell \geq 2$ and $0 < \alpha < 1$. Assume that*

(a) Ω is a bounded domain in R^n, $n \geq 2$, $\partial\Omega \in C^{\ell+2+\alpha} \cap K_m$;

(b) $\varphi \in C^{\ell+2+\alpha}(\partial\Omega)$;

(c) $H_m(x,u) \in C^{\ell+\alpha}(\bar{\Omega} \times R^1)$, $\partial H_m/\partial u \geq 0$, $H_m(x,u) \geq \nu > 0$, and

$$\int_\Omega H_m^{n/m}(x,\varphi_1)\,dx \leq (1-\chi)\omega_n \tag{10}$$

with some $\chi > 0$, ω_n being the volume of the unit ball in R^n;

(d) the m-curvature of $\partial\Omega$ denoted by $h_m(x)$ is connected with $H_m(x,\varphi_1)$ by the inequality

$$H_m(x,\varphi_1) \leq \frac{n-m}{n}\,h_m(x)\,, \quad x \in \partial\Omega\,. \tag{11}$$

Then there exists a unique solution $u \in C^{\ell+2+\alpha}(\bar{\Omega}) \cap K_m$ of the problem

$$\mu_m[u] = H_m(x,u)\,, \quad u\big|_{\partial\Omega} = \varphi(x)\,. \tag{12}$$

Corollary. *There exists a set $\mathcal{M}_u$ such that the functional I_{m-1} achieves its local minimum on $\mathcal{M}_u$ if the assumptions of theorem 4 are fulfilled for some $\ell \geq 2$.*

Remarks. (i) The inequality (10) was formulated in [4] as a consequence of a sharp condition:

$$\int_E H_m(x,\varphi_1)\,dx \leq (1-\chi)\int_{\partial E} \mu_{m-1}(\partial E)\,ds$$

where E is any subdomain of Ω with $\partial E \in K_{m-1}$ (see [4]).

(ii) The inequality (11) was discovered by Trudinger [5], [6] as being necessary for solvability of problem (12) with any smooth boundary function $\varphi(x)$.

Theorem 4 may be proved by combining some results from [4]–[6], [7], [8].

REFERENCES

[1] N.M. Ivochkina, Variational problems connected with operators of Monge-Ampère type, *Zap. Nauch. Semin. LOMI* **167** (1988), 186–189.

[2] N.M. Ivochkina, A description of the stability cones generated by differential operators of Monge-Ampère type, *Mat. Sb. (N.S.)* **122** (1983), 265–275 (Russian). English translation in *Math. USSR Sb.* **50** (1985), 259–268.

[3] N.M. Ivochkina, Quasivariational problems connected with m-curvature equations, Preprint CMA (1989).

[4] N.S. Trudinger, A priori bounds for graphs with prescribed curvature, *Festschrift for Jürgen Moser*, Academic Press, 1989.

[5] N.S. Trudinger, The Dirichlet problem for the prescribed curvature equations, Preprint CMA-R19-89.

[6] N.S. Trudinger, A priori bounds for solutions of prescribed curvature equations, Preprint CMA (1989).

[7] L. Caffarelli, L. Nirenberg and J. Spruck, Nonlinear second order elliptic equations V. The Dirichlet problem for Weingarten hypersurfaces, *Comm. Pure Appl. Math.* **41** (1988), 47–70.

[8] N.M. Ivochkina, Solution of the Dirichlet problem for an equation of curvature of order m, *DAN USSR* **299** (1988), 35–38 (Russian).

[9] N.M. Ivochkina, The Dirichlet problem for an equation of curvature of order m, *Algebra and Analysis* **6** (1989) (Russian).

AMS SUBJECT CLASSIFICATION NOS: 35J60 35J65

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