AN EXTENSION OF A THEOREM OF HALMOS TO ORLICZ SPACES, AND AN APPLICATION TO NEMITSKY OPERATORS

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1. INTRODUCTION

In our paper [3], we gave an extension to Orlicz-Sobolev spaces of a theorem of Marcus and Mizel (Theorem 2.1 of [5]) on the mapping of Sobolev spaces by Nemitsky Operators. However we did not give an extension of their theorem on demicontinuity (i.e., "strong → weak" continuity) of Nemitsky operators - Theorem 4.1 in [5]. Essentially this was because Marcus and Mizel, in their proof, used a theorem of Halmos (see [2]) on the mapping of Lebesgue spaces under composition by Borel measurable functions. We did not have a suitable (i.e., compatible with the hypotheses made in [3]) version available for Orlicz spaces. We present here the needed extension of Halmos' theorem to Orlicz spaces, and an extension, to Orlicz-Sobolev spaces, of Marcus and Mizel's theorem on demicontinuity.

2. DEFINITIONS

Throughout this section, Ω denotes a bounded domain in $\mathbb{R}^n$.

(i) Orlicz Spaces

We shall only give the definitions needed to set the notation which occurs in the statements of our theorems. Further details and references (for (ii) below also) can be found in [1].

An *N-function* is a real valued continuous convex even function on $\mathbb{R}$, such that $M(u)/u \to 0(\infty)$ as $u \to 0(\infty)$. The *complementary* N-function $\tilde{M}$ to M is defined by $\tilde{M}(v) = \sup_{u \geq 0} [u \mid v \mid - M(u)]$. We say that M satisfies the Δ_2-*condition* if there exists

$k > 0$ such that $M(2u) \leq kM(u)$ for large u, and given two N-functions M_1 and M_2, we write $M_1 \prec M_2$ if there exists $k > 0$ such that $M_1(u) \leq M_2(ku)$ for large u. We define the *Orlicz class* and *space* respectively by

$$L^*_M(\Omega) = \{u : \int_\Omega M \circ u < \infty\},$$

and

$$L_M(\Omega) = \{u : \text{there exists } k > 0 \text{ such that} \int M[ku(x)]dx < \infty\},$$

(u denotes a function measurable on Ω) and we define a norm on $L_M(\Omega)$ by

$$\| u \|_{M,\Omega} = \inf\{\lambda > 0 : \int_\Omega M[u(x)/\lambda]dx \leq 1\}.$$

For a vector-valued function $u = (u_1, \cdots, u_m)$ and N-functions $Q_1, \cdots, Q_m$, we write

$$\| u \|_{\overline{Q},\Omega} = \max_i \| u \|_{Q_i,\Omega}.$$

(ii) Orlicz-Sobolev Spaces

The *Orlicz-Sobolev space* $W^1L_Q(\Omega)$ is defined as the set of all functions u in $L_Q(\Omega)$ whose distributional derivatives $\partial_{x_i} u$ also belong to $L_Q(\Omega)$, with norm $\| u \|_{1,Q,\Omega} = \max\{\| u \|_Q, \| \partial_{x_1} u \|_Q, \cdots, \| \partial x_n u \|_Q\}$. The *Sobolev conjugate* N-function Q^* of an N-function Q is defined by

$$(Q^*)^{-1}(s) = \int_0^{|s|} Q^{-1}(t)t^{-1-1/n}\, dt$$

with Q redefined, if necessary, for small t so that $(Q^*)^{-1}(1) < \infty$.

(iii) Caratheodory Functions and Nemitsky Operators

Let $A(\Omega)$ denote the class of real measurable functions u on Ω such that, for almost every line τ parallel to any co-ordinate axis, u is locally absolutely continuous on $\tau \cap \Omega$. $A'(\Omega)$ denotes the class of functions u such that u coincides almost everywhere in Ω with a function $\tilde{u}$ in $A(\Omega)$. For $u \in A'(\Omega)$, the symbol $\partial'_{x_j} u$ denotes any member of the equivalence class of functions measurable on Ω which contains the classical partial derivative $\partial \tilde{u} / \partial x_j$.

A function $g : \Omega \times \mathbf{R}^m \to \mathbf{R}$ is said to be a *locally absolutely continuous Caratheodory function* if

(i) there exists a null subset N_g of Ω such that if $x \in \Omega - N_g$,

  (a) $g(x, \cdot)$ is continuous in $\mathbf{R}^m$,

  (b) for every line τ parallel to one of the axes in $\mathbf{R}^m$, the function $g(x, \cdot)$ restricted to this line is locally absolutely continuous;

(ii) for every fixed $t \in \mathbf{R}^m$ we have $g(\cdot, t) \in A'(\Omega)$.

An operator G on vector valued functions $u = (u_1, \cdots, u_m)$ measurable on Ω defined by

$$Gu(x) = g(x, u(x))$$

is called a *Nemitsky operator.*

3. TWO THEOREMS

Our version of Halmos' theorem is:

THEOREM 1. *Let f be a real valued Borel measurable function on $\mathbf{R}^m$, and Ω be a bounded domain in $\mathbf{R}^n$. Let $P_1, \cdots, P_m$ and Q be N-functions.*

(a) If $f(u_1, \cdots, u_m) \in L_Q(\Omega)$ for all $u_k \in L^*_{P_k}(\Omega)$, $k = 1, \cdots, m$, then there exists a constant C such that, for all points $(\sigma_1, \cdots, \sigma_m)$ in $\mathbf{R}^m$, the inequality

(i) $$| f(\sigma_1, \cdots, \sigma_m)| \leq C \sum_{k=1}^{m} Q^{-1} \circ P_k(1 + |\sigma_k|)$$

is satisfied.

(b) If f satisfies an inequality of the form (i) for all $(\sigma_1, \cdots, \sigma_m) \in \mathbf{R}^m$, and if each function $Q^{-1} \circ P_k, k = 1, \cdots, m$, satisfies the Δ_2-condition, then $f(u_1, \cdots, u_m) \in L_Q(\Omega)$ whenever $u_k \in L_{P_k}(\Omega), k = 1, \cdots, m$.

Using the above, routine modifications of Marcus & Mizels arguments yield the following version of their theorem on demicontinuity:

THEOREM 2. *Let Ω be a bounded domain in $\mathbf{R}^n$ having the cone property, and let g be a locally absolutely continuous Caratheodory function on $\Omega \times \mathbf{R}^m$, Let P, Q_k,* and *$Q^{\dagger}_k$,* for *$k = 1, \cdots, m$, be N-functions having the following properties:*

(i) P, $\tilde{P}$, and Q_k satisfy the *Δ_2-condition;*

(ii) $P < Q_k$;

(iii) there exist complementary N-functions R_k and *$\tilde{R}_k$ such that*

$$R_k \prec P^{-1} \circ Q_k$$

and

$$\tilde{R}_k \prec P^{-1} \circ Q^{\dagger}_k$$

(iv) $(Q^{\dagger}_k)^{-1} \circ Q^*_j$ for *$k, j = 1, \cdots, m$,* and *$P^{-1} \circ Q^*_j$* for *$j = 1, \cdots, m$, satisfy the Δ_2-condition;*

(v) $(Q^*_k)^{-1}(v) \to \infty$ as $v \to \infty$.

Suppose a, b, a_k and *$b_{k,j}$ are functions such that*

I *For every fixed* $t \in \mathbb{R}^m$,

$$|\partial'_{x_i} g(x,t)| \leq a(x) + b(t) \text{ almost everywhere in } \Omega, \text{ for } i = 1, \cdots, n .$$

II *The inequality*

$$|\partial g(x,t)/\partial t_k| \leq a_k(x) + \sum_{j=1}^{m} b_{k,j}(t_j), \text{ for } k = 1, \cdots, m$$

holds at every point $(x,t) \in (\Omega - N_g) \times \mathbb{R}^m$ *at which the derivative exists in the classical sense.*

Furthermore, a, b, a_k and $b_{k,j}$ *have the properties (vi) to (xiii) given below:*

(vi) $0 \leq a \in L_P(\Omega)$;

(vii) b *is non-negative and continuous in* $\mathbb{R}^m$;

(viii) $0 \leq a_k \in L_{Q^\dagger_k}(\Omega)$, $k = 1, \cdots, m$, *where* a_k *is everywhere finite;*

(ix) $0 \leq b_{k,j}$ *is an everywhere finite Borel function on* $\mathbb{R}$, $k, j = 1, \cdots, m$;

(x) $b_{k,k}$ *is continuous*, $k = 1, \cdots, m$;

(xi) b *defines, by composition, a mapping from* $L_{Q^*_1}(\Omega) \times \cdots \times L_{Q^*_m}(\Omega)$ *to* $L_P(\Omega)$;

(xii) $b_{k,j}$ *defines, by composition, a mapping from* $L_{Q^*_j}(\Omega)$ *to* $L_{Q^\dagger_k}(\Omega)$, *for* $k, j = 1, \cdots, m$;

(xiii) the mappings in (xi) and, for $j = k$ *in (xii), are continuous.*

Then G *maps* $W^1L_{\overline{Q}}(\Omega)$ *into* $W^1L_P(\Omega)$ *and is demicontinuous and bounded. Moreover* G *is continuous as a mapping from* $W^1L_{\overline{Q}}(\Omega)$ *to* $L_P(\Omega)$.

Example. Let $1 < p < n$. Then the family of N-functions defined by $P(u) = |u|^p$, $R_k(u) = (1+|u|)\ln(1+|u|) - |u|$, $\tilde{R}_k(u) = e^{|u|} - |u| - 1$,

$Q_k(u) = [1+|u|)\ln(1+|u|) - |u|]^p$ and $Q^{\dagger}_k(u) = [e^{|u|} - |u| - 1]^p$, satisfies the hypotheses of Theorem 2. It may be shown that, in this case, the hypothesis (xiii) is unnecessary.

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