A General Markus Inequality

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An inequality of A.S. Markus for the singular values of compact operators is generalized to measurable operators; this inequality is then used as a tool to construct a wide class of rearrangement invariant Banach spaces of measurable operators.

0. Introduction

In this note we wish to outline a method of construction of a wide class of Banach spaces of operators which are the non-commutative analogues of the rearrangement invariant Banach function spaces studied, for example, in Luxemburg [Lu], Fremlin [Fr] and Krein, Petunin and Semenov [KPS]. We gather first some convenient notation.

If x is a compact operator on the Hilbert space $\mathcal{H}$ then the real sequence $\{\mu_n(x) : n = 0, 1, 2, \cdots\}$ of eigenvalues of $|x| = (x^*x)^{1/2}$, arranged in decreasing order and repeated according to multiplicity is called the *singular value sequence* of x. If, for any bounded real sequence $\{a_n\}_{n=0}^{\infty}$ we denote by $\{a_n^*\}_{n=0}^{\infty}$ the decreasing rearrangement of the sequence $\{|a_n|\}_{n=0}^{\infty}$, then we say that the sequence $\{b_n\}_{n=0}^{\infty}$ is *submajorized* by the sequence $\{a_n\}_{n=0}^{\infty}$ (in the sense of Hardy, Littlewood and Polya), and we write

$$\{b_n\}_{n=0}^{\infty} \prec\prec \{a_n\}_{n=0}^{\infty},$$

if and only if

$$\sum_{j=0}^{k} b_j^* \leq \sum_{j=0}^{k} a_j^*, \qquad \text{for } k = 0, 1, 2, \cdots.$$

Suppose now that x, y are compact operators on the Hilbert space $\mathcal{H}$. It is a well-known result of Markus [Ma] that

$$\{\mu_n(x) - \mu_n(y)\}_{n=0}^{\infty} \prec\prec \{\mu_n(x-y)\}_{n=0}^{\infty}.$$

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An analogous result concerning decreasing rearrangements of measurable functions is due to Lorentz and Shimogaki [LS]. The basis of our approach to the construction of non-commutative Banach function spaces is to show that the inequalities of Markus and of Lorentz - Shimogaki admit a common generalization to arbitrary measurable operators (in the sense of Nelson [Ne]) affiliated with a semi-finite von Neumann algebra, via the notion of generalized singular value studied by Ovčinnikov [Ov1], Yeadon [Ye1] and Fack [Fa]. This general Markus inequality permits the application of standard methods to prove the triangle inequality and norm completeness of a class of non-commutative Banach function spaces which seems to be somewhat more general than has been so far considered in the literature. Let us mention that, for the case of finite trace, a similar result has been obtained by F.A. Sukochev [Su1], [Su2]. We take here the opportunity of thanking V.I. Chilin, F.A. Sukochev, A.V. Krygin and A.M. Medzhitov for kindly communicating their results on the present as well as related themes and, in particular for pointing out, at least implicitly, the sharpening of [DDP1] Theorem 4.5 that is outlined in Theorem 2.1 below.

1. A general Markus Inequality

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ be a von Neumann algebra with a normal faithful semifinite trace τ. A closed densely defined linear operator x in $\mathcal{H}$ is said to be *affiliated with* $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$ then x is called *τ-measurable* if and only if, there exists a number $s \geq 0$ such that

$$\tau(\chi_{(s,\infty)}(|x|)) < \infty.$$

We denote by $\widetilde{\mathcal{M}}$ the set of all τ-measurable operators. Sum and product in $\widetilde{\mathcal{M}}$ are defined as the respective closures of the algebraic sum and product. For $x \in \widetilde{\mathcal{M}}$, the *generalized singular value function* $\mu_{.}(x)$ of x is defined by

$$\mu_t(x) = \inf\{s \geq 0 : \tau(\chi_{(s,\infty)}(|x|)) \leq t\}, \qquad t \geq 0.$$

It follows simply that $\mu_{.}(x)$ is a decreasing, right-continuous function on the half line

$[0,\infty)$.

The idea of measurable operators with respect to a trace goes back to I. Segal [Se]. The above definition of measurability is due to E. Nelson [Ne] .While the class of measurable operators defined by Nelson is in general a proper subset of Segal's class, it is the natural setting in which to consider properties related to generalized singular values.

We define a translation invariant metric on $\widetilde{\mathcal{M}}$ by setting

$$d(x,0) = \inf\{t \geq 0 : \mu_t(x) \leq t\}, \qquad \text{for } x \in \widetilde{\mathcal{M}},$$

and

$$d(x,y) = d(x-y,0), \qquad \text{for } x,y \in \widetilde{\mathcal{M}}.$$

The topology defined by d is called the *measure topology.* It is not difficult to see that the measure topology has a neighbourhood basis of zero, consisting of all sets of the form

$$N(\epsilon,\delta) = \{x \in \widetilde{\mathcal{M}} : \exists \text{ projection } e \in \mathcal{M} \text{ such that } \|xe\| \leq \epsilon \text{ and } \tau(1-e) \leq \delta\}$$

where ϵ, δ are positive numbers, which is the definition given by Nelson [Ne]. It is shown in [Ne] and [Te] that $\widetilde{\mathcal{M}}$ equipped with the measure topology is a complete, Hausdorff, topological *-algebra in which $\mathcal{M}$ is dense.

For the convenience of the reader, we gather here some of the basic properties of generalized singular values. The proofs of most of these properties can be found variously in Ovčinnikov [Ov1], Yeadon [Ye1], Fack and Kosaki [FK]. We follow [FK] Lemma 2.5.

Proposition 1.1. *Let* $x,y \in \widetilde{\mathcal{M}}$.

(a) The singular value function $\mu(x)$ *admits the characterization*

$$\mu_t(x) = \inf\{\|xe\|_\infty : e \text{ is a projection in } \mathcal{M}, \tau(1-e) \leq t\}, \qquad t > 0.$$

(b) $\lim_{t\to 0+} \mu_t(x) = \|x\|_\infty \in [0,\infty]$.

(c) For each $t > 0$ *and* $\alpha \in \mathbb{C}$,

$$\mu_t(x) = \mu_t(|x|) = \mu_t(x^*)$$
$$\mu_t(\alpha x) = |\alpha|\mu_t(x).$$

(d) The equality

$$\mu_t(\phi(|x|)) = \phi(\mu_t(|x|))$$

holds for each $t > 0$ *and any continuous increasing function* ϕ *on* $[0, \infty)$ *with* $\phi(0) \geq 0$.

(e) $\mu_{t+s}(x+y) \leq \mu_t(x) + \mu_s(y), \qquad t, s > 0.$

(f) $\mu_t(uxv) \leq \|u\|_\infty \|v\|_\infty \mu_t(x), \qquad t > 0,\ u, v \in \mathcal{M}.$

(g) $\mu_{t+s}(xy) \leq \mu_t(x)\mu_s(y), \qquad t, s > 0.$

We now consider the partial ordering defined on $\widetilde{\mathcal{M}}$ by setting

$$x \geq 0 \text{ if and only if } < x\xi, \xi > \geq 0$$

for all ξ in the domain of x. This partial order may be shown to have the following properties [DDP2], [FK].

Proposition 1.2. *(a)* $(\widetilde{\mathcal{M}}, \leq)$ *is an ordered vector space and the positive cone* $\widetilde{\mathcal{M}}_+$ *is closed in* $\widetilde{\mathcal{M}}$ *for the measure topology.*

(b) $\widetilde{\mathcal{M}}$ *is order complete in the sense that every increasing order bounded net in the positive cone of* $\widetilde{\mathcal{M}}$ *has a supremum in* $\widetilde{\mathcal{M}}$.

(c) If $\{x_\alpha\}$ *is an increasing net in* $\widetilde{\mathcal{M}}_+$ *and if*

$$x = sup\ x_\alpha$$

holds in $\widetilde{\mathcal{M}}$, *then*

$$\mu(x_\alpha) \uparrow_\alpha \mu(x)$$

holds in the space of all Lebesgue measurable functions on $[0, \infty)$.

(d) The trace τ *can be extended to the positive cone* $\widetilde{\mathcal{M}}_+$ *and this extension*

$$\tau : \widetilde{\mathcal{M}}_+ \longrightarrow [0, \infty]$$

is additive, positive homogeneous, unitarily invariant and normal.

(e) The equality

$$\tau(\phi(|x|)) = \int_{[0,\infty)} \phi(\mu_t(x))dt, \quad x \in \widetilde{\mathcal{M}}$$

holds whenever ϕ *is a continuous strictly increasing function on* $[0,\infty)$ *with* $\phi(0) = 0$.

It is now worth pausing to consider three special examples.

Example 1.3. If $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and τ is the standard trace, then $\widetilde{\mathcal{M}} = \mathcal{M}$ and the measure topology coincides with the operator norm topology. If $x \in \mathcal{M}$ is compact, then it is not difficult to see that, for each $n = 0, 1, 2, \cdots$,

$$\mu_n(x) = \mu_t(x), \qquad n \leq t < n+1,$$

and $\{\mu_n(x)\}_{n=0}^{\infty}$ is the sequence of eigenvalues of $|x|$, in decreasing order counted according to multiplicity. Of course, in the setting of this example, the assertions of Proposition 1.1 (a) reduce to well known consequences of the Courant-Fischer minimax characterizations for the singular values of compact operators as given, for example in the monograph of Gohberg and Krein [GK].

Example 1.4. If the trace τ is finite, then $\widetilde{\mathcal{M}}$ coincides with the set of all closed densely defined linear operators affiliated with $\mathcal{M}$.

Example 1.5. If $\mathcal{M}$ is commutative, then $\mathcal{M}$ can be identified as $L^\infty(\Omega, \Sigma, \nu)$ for some measure space (Ω, Σ, ν) with a localizable measure ν, acting by multiplication on the Hilbert space $L^2(\Omega, \Sigma, \nu)$. If the trace τ is defined by setting

$$\tau(f) = \int_\Omega f d\nu, \qquad 0 \leq f \in \mathcal{M},$$

then $\widetilde{\mathcal{M}}$ is the set of all ν-measurable functions bounded except on a set of finite measure, and, the measure topology is the topology of convergence in measure. In this case, the generalized singular value function $\mu(x)$, $x \in L^\infty(\Omega, \Sigma, \nu)$ is given by

$$\mu_t(x) = \inf\{s \geq 0 : \nu(\{\omega \in \Omega : |x(\omega)| > s\}) \leq t\}, \qquad t > 0$$

so that the singular value function $\mu(x)$ coincides with the classical non-increasing rearrangement of x. In this case, the results of Proposition 1.1 are familiar from Luxemburg [Lu], Fremlin [Fr] or Krein-Petunin-Semenov [KPS].

Before proceeding to the main result of this section, we introduce a notion of submajorization which goes back to Hardy, Littlewood and Polya.

Definition 1.6. For $x, y \in \widetilde{\mathcal{M}}$, we say that x is *submajorized by* y, written $x \prec\prec y$, if and only if

$$\int_0^\alpha \mu_t(x)dt \leq \int_0^\alpha \mu_t(y)dt, \qquad \forall \alpha \geq 0.$$

We may now state the principal result of this section.

Theorem 1.7. *If $x, y \in \widetilde{\mathcal{M}}$, then*

$$\mu(x) - \mu(y) \prec\prec \mu(x-y)$$

(where the submajorization is taken in the von Neumann algebra $L^\infty([0,\infty))$ as in Example 1.5). Equivalently,

$$\sup_{|E| \leq \alpha} \int_E |\mu_t(x) - \mu_t(y)| dt \leq \int_0^\alpha \mu_t(x-y)dt, \qquad \forall \alpha \geq 0$$

where $|E|$ is the Lebesgue measure of the set E.

The details of proof of Theorem 1.7 are given in [DDP1]. We note that Theorem 1.7 is due to A.M. Markus [Ma] for the case that $\mathcal{M} = \mathcal{L}(\mathcal{H})$, τ the standard trace and x, y compact. The method of Markus is based on an older result of Wielandt and the method of [DDP1] is an extension of that of Markus. For the case that the trace τ is finite, Theorem 1.7 was also proved by F. Hiai and Y. Nakamura [HN] by a different method. Finally, let us observe that Theorem 1.7 is due to G.G. Lorentz and T. Shimogaki [LS] in the case that $\mathcal{M}$ is commutative.

2. Non-commutative Banach function spaces

Let $L^0(\mathbb{R}^+)$ be the linear space of all (equivalence classes of) real-valued Lebesgue measurable functions on the half line $[0,\infty)$. A Banach space E which is a non-zero linear subspace of $L^0(\mathbb{R}^+)$, is called a *Banach function space* on the half line $\mathbb{R}^+$, if and only if,

$$f \in E, g \in L^0(\mathbb{R}^+) \text{ and } |g| \leq |f| \implies g \in E \text{ and } \|g\|_E \leq \|f\|_E.$$

Let $E \subseteq L^0(\mathbb{R}^+)$ be a Banach function space on $\mathbb{R}^+$. We say that E is *rearrangement invariant* if and only if

$$f \in E, g \in L^0(\mathbb{R}^+), \mu(g) \leq \mu(f) \implies g \in E \text{ and } \|g\|_E \leq \|f\|_E.$$

We say that E is *symmetric*, if and only if

$$f, g \in E \text{ and } \mu(g) \prec\prec \mu(f) \implies \|g\|_E \leq \|f\|_E.$$

If E is a rearrangement invariant Banach function space on $\mathbb{R}^+$, we define

$$E(\widetilde{\mathcal{M}}) = \{x \in \widetilde{\mathcal{M}} : \mu(x) \in E\}$$

and set

$$\|x\|_E = \|\mu(x)\|_E, \qquad x \in E(\widetilde{\mathcal{M}}).$$

Our principal result on the construction of non-commutative Banach function spaces now follows.

Theorem 2.1. *If E is a symmetric rearrangement invariant Banach function space on $\mathbb{R}^+$, then $\|.\|_E$ defines a norm on $E(\widetilde{\mathcal{M}})$ and $(E(\widetilde{\mathcal{M}}), \|.\|_E)$ is a Banach space.*

Outline of proof.

From the inequality

$$\mu_t(x+y) \leq \mu_{\frac{t}{2}}(x) + \mu_{\frac{t}{2}}(y), \qquad \forall t \geq 0$$

and the fact that E is closed under dilation, it follows that $E(\widetilde{\mathcal{M}})$ is a linear subspace of $\widetilde{\mathcal{M}}$. The triangle inequality follows by observing that

$$\mu(x+y) \prec\prec \mu(x) + \mu(y), \qquad \forall x, y \in E(\widetilde{\mathcal{M}}),$$

which is a consequence of Theorem 1.7.

To show norm completeness, let $\{x_n\}$ be a Cauchy sequence in $E(\widetilde{\mathcal{M}})$. Since the natural inclusion of $(E(\widetilde{\mathcal{M}}), \|.\|)$ into $(\widetilde{\mathcal{M}}, d)$ is continuous and $(\widetilde{\mathcal{M}}, d)$ is complete, there exists $x \in \widetilde{\mathcal{M}}$ such that

$$x_n \longrightarrow x \qquad \text{in} \qquad (\widetilde{\mathcal{M}}, d),$$

which implies that

$$\mu(x_n) \longrightarrow \mu(x) \qquad a.e..$$

On the other hand, the general Markus inequality implies that

$$|\mu(x_n) - \mu(x_m)| \prec\prec \mu(x_n - x_m) \qquad \forall n, m,$$

and since the norm on E is symmetric, it follows that the sequence $\{\mu(x_n)\}$ is Cauchy in E, and hence convergent in E. From this it follows that $\mu(x) \in E$ and consequently $x \in E(\widetilde{\mathcal{M}})$.

It remains to show that

$$\|x - x_n\|_E \longrightarrow 0 \qquad \text{as} \qquad n \to \infty.$$

Again using the general Markus inequality, and the fact that E is symmetric it follows that for each $n = 1, 2, \cdots$,

$$\|\mu(x_k - x_n) - \mu(x_l - x_n)\|_E \leq \|\mu(x_k - x_l)\|_E \to 0 \qquad \text{as} \qquad k, l \to \infty.$$

Now

$$\|\mu(x - x_n)\|_E \leq \|\mu(x - x_n) - \mu(x_l - x_n)\|_E + \|\mu(x_l - x_n)\|_E,$$

hence

$$\|x - x_n\|_E = \|\mu(x - x_n)\|_E \longrightarrow 0 \qquad \text{as} \qquad n \to \infty,$$

and the proof is complete.

For the case that the trace τ is finite, the preceding Theorem 2.1 has been proved by F. Sukochev [Su1], [Su2]. The proof given above is almost identical to that given in [DDP1] Theorem 4.5, where it is assumed that the norm on E is lower-semicontinuous with respect to pointwise sequential convergence in E (or a Fatou norm in the terminology of [Za]). This property however implies that E is symmetric, via a well-known theorem of G.G. Lorentz and W.A.J. Luxemburg and consequently Theorem 2.1 preceding in fact sharpens the result, if not the proof, of [DDP1] Theorem 4.5. To indicate the scope of applicability of Theorem 2.1, it suffices to point out that the hypotheses of Theorem 2.1 (indeed even the hypotheses of [DDP1] Theorem 4.5) are readily seen to be satisfied, for example, by the familiar Orlicz spaces, for the Lorentz and Marcinkiewicz spaces as defined in [KPS], [Ca] or for the class of (maximal) Köthe spaces given in Ovčinnikov [Ov2]. All of these spaces as well as those considered by Yeadon [Ye2] are, in addition, interpolation spaces for the Banach couple $(L^1(\mathbb{R}^+), L^\infty(\mathbb{R}^+))$. It is therefore not without interest to point out that the present approach includes as well nontrivial examples of rearrangement invariant symmetric function spaces on the half-line which are not interpolation spaces such as that given by [KPS] Theorem 5.11.

It should also be observed that if E is one of the familiar L^p-spaces, $1 \leq p < \infty$, then the spaces $L^p(\widetilde{\mathcal{M}})$ given by the preceding construction coincide with those defined by Nelson [Ne]; indeed, this observation is a consequence of Proposition 1.2(e). In addition, it is easy to see that the equality $L^\infty(\widetilde{\mathcal{M}}) = \mathcal{M}$ holds with equality of norms.

We mention finally that it has been shown, for example in [Ov2], [PS] and [FK] that the equality

$$(L^1 + L^\infty)(\widetilde{\mathcal{M}}) = L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$$

holds with equality of norms. As shown in [KPS], Theorem II4.1, any rearrangement invariant Banach function space E on the half-line $[0, \infty)$ is intermediate for the Banach couple $(L^1(\mathbb{R}^+), L^\infty(\mathbb{R}^+))$. It follows immediately that if in addition E is symmetric, then the space $E(\widetilde{\mathcal{M}})$ is intermediate for the Banach couple $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$.

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