Measures of Semi-noncompactness and AM-mappings

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In this paper, we define measures of semi-noncompactness in a locally convex topological linear space with respect to a given seminorm, and give some simple properties, including a fixed point theorem for a certain class of condensing mappings.

0. Introduction.

Generalizations of the notion of a completely continuous mapping which are related to the study of measures of non-compactness and which have found fruitful application in nonlinear analysis have been given by Darbo [4] and Nussbaum [10]. More recently, de Pagter and Schep [11] have studied a measure of non-compactness for positive operators on Banach lattices which exploits the underlying order structure and its relation to the norm topology. In this paper, motivated by the ideas of [11], we introduce a notion of measure of semi-noncompactness for subsets of a topological vector lattice which not only reduces to that given in [11] for the norm topology on a Banach lattice, but which applies equally to the weak topology in a wide class of Banach lattices. These ideas lead naturally to considering a class of nonlinear mappings which we call AM-mappings and for which we prove a fixed point theorem (Theorem 3). An important example of an AM-mapping is given by the (so-called) Nemytsky, or substitution, operator on the Banach lattice $L^1(0,1)$ associated with a given Carathéodory function and we illustrate our ideas by giving a proof of a recent result of Banaś [2] which highlights the natural role played by the order structure (Theorem 4). We assume throughout that the reader has some familiarity with the terminology and theory of Banach lattices, as can be found in the books [7], [9], [12].

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1.Definitions and basic results

Let (E, τ) be a real, locally convex topological linear space. We assume that there is an order relation $\leq$ in E, which makes E a vector lattice. For $x \in E$, let $x^+ = x \vee 0, x^- = (-x) \vee 0, |x| = x^+ + x^-$, and $E_+ = \{ x \in E |\ x \geq 0\}$. We assume that the topology τ and the partial order $\leq$ satisfy the following condition (H):

(H). If $x \in E$ and if $\{x_n\} \subset E$ is any sequence which is $\tau-$convergent to x, then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and there exist elements $y, z \in E$ with $0 \leq x^+ \leq y$, $0 \leq x^- \leq z$ such that the subsequence $\{x^+_{n_j}\}$ (respectively $\{x^-_{n_j}\}$) is τ-convergent to y (respectively z).

It is clear that condition (H) implies that the positive cone E_+ is τ-closed in E. We remark that if E is a Banach lattice and if τ is the norm topology on E then condition (H) is clearly satisfied since the lattice operations are continuous for the norm topology. If τ is the weak topology on a Banach lattice then the situation is somewhat different, since that lattice operations are, in general, not weakly continuous. However, condition (H) will be satisfied for the weak topology of a Banach lattice E if the solid hull of any weakly compact subset of E is again relatively weakly compact, in particular, if E is reflexive or if E is an abstract L-space.

Let ϕ be a seminorm in E, which is lower semicontinuous with respect to τ, that is, if $B_\phi = \{x \in E |\ \phi(x) \leq 1\}$, then B_ϕ is τ-closed. In addition, we suppose that ϕ is monotone with respect to the given partial order, that is, $0 \leq x \leq y$ implies $\phi(x) \leq \phi(y)$. If $D \subset E$, and if there exists $r > 0$ such that $D \subset rB_\phi$, then D is called $\phi-$bounded. D is called *almost order-bounded relative to* ϕ, if given $\epsilon > 0$, there exists $u \in E_+$ such that $D \subset [-u, u] + \epsilon B_\phi$. This is equivalent to the statement: given $\epsilon > 0$, there exists $u \in E_+$ such that

$$\phi((|x| - u)^+) \leq \epsilon, \qquad \forall x \in D.$$

We remark that if E is an abstract L-space and if ϕ is the given norm on E then a subset $D \subset E$ is almost order-bounded relative to ϕ if and only if D is relatively weakly compact.

See, for example, [7]. For any $\phi-$bounded subset D in E, define

$$\rho_\phi(D) = \inf\{\delta > 0 \mid \exists u \in E_+ \text{ such that } D \subset [-u,u] + \delta B_\phi \}.$$

It is easily seen that

$$\rho_\phi(D) = \inf\{\delta > 0 \mid \exists u \in E_+ \text{ such that } \phi((|x| - u)^+) \leq \delta, \forall x \in D \}.$$

We say that $\rho_\phi(D)$ is the measure of semi-noncompactness of D with respect to ϕ. We will omit ϕ if there is no danger of confusion. Our definition is motivated by the measure of semi-noncompactness introduced by de Pagter and Schep [11] and reduces to theirs for the case that E is a Banach lattice with ϕ the given norm on E and τ the norm topology. Morover, if E is an abstract L-space and if ϕ is the given norm on E then ρ_ϕ is the measure of weak non-compactness introduced by F.S.de Blasi [3].

We now gather some simple properties, which for the case that E is a Banach lattice and τ is the norm topology on E may be found in [3].

Lemma 1. *If D, D_1, D_2 are ϕ-bounded sets in E, then*

(i). $\rho(D) = 0 \iff D$ *is almost order-bounded (relative to ϕ);*

(ii). $\rho(D_1 + D_2) \leq \rho(D_1) + \rho(D_2), \quad \rho(\lambda D) = |\lambda|\rho(D), \quad \lambda$ *real;*

(iii). $D_1 \subset D_2 \Rightarrow \rho(D_1) \leq \rho(D_2)$*;*

(iv). $\rho(D \bigcup \{x_0\}) = \rho(D), \quad x_0 \in E$*;*

(v). $\rho(\overline{D}) = \rho(D)$*, where $\overline{D}$ denotes the closure of D with respect to τ;*

(vi). $\rho(\overline{co}(D)) = \rho(D)$*, where $\overline{co}(D)$ is $\tau-$convex closure of D.*

Proof. (i), (iii) and (iv) are clear.

(ii) Let $x_k \in D_k, \; k = 1,2$. Given $\epsilon > 0$, there exists $u_k \geq 0$ such that

$$D_k \subset [-u_k, u_k] + (\rho(D_k) + \frac{\epsilon}{2})B_\phi, \quad k = 1,2.$$

If $x_k = y_k + z_k$, with $y_k \in [-u_k, u_k]$ and $z_k \in (\rho(D_k) + \frac{\epsilon}{2})B_\phi$, $k = 1, 2$, then

$$x_1 + x_2 \in [-(u_1 \vee u_2), u_1 \vee u_2] + (\rho(D_1) + \rho(D_2) + \epsilon)B_\phi.$$

It follows that

$$\rho(D_1 + D_2) \leq \rho(D_1) + \rho(D_2) + \epsilon,$$

for every $\epsilon > 0$, and (ii) follows.

(v) From (iii), it follows that $\rho(D) \leq \rho(\overline{D})$. To prove the reverse inequality, if $\epsilon > 0$ is given, then there exists $u \in E_+$ such that

$$\phi((|x| - u)^+) \leq \rho(D) + \epsilon, \qquad \forall x \in D.$$

If $x \in \overline{D}$,there exsists a sequence $\{x_n\} \subset D$, with $x_n \to_\tau x$. By (H), there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ satisfying: $x_{n_j}^+ \to y$, $x_{n_j}{}^+ \to z$, and $x^+ \leq y$, $x^- \leq z$. So $|x_{n_j}| - u \to_\tau v - u$, where $v = y + z$. By (H) again, there exists a subsequence $\{x_{n_{j'}}\}$ of $\{x_{n_j}\}$ such that

$$(|x| - u)^+ \leq (v - u)^+ \leq \lim_{j'}(|x_{n_{j'}}| - u)^+.$$

From lower semicontinuity of ϕ relative to τ and the fact ϕ is monotone, we have

$$\begin{aligned} \phi((|x| - u)^+) &\leq \phi((v - u)^+) \\ &\leq \phi(\lim_{j'}(|x_{n_{j'}}| - u)^+) \\ &\leq \liminf_{j'} \phi((|x_{n_{j'}}| - u)^+) \\ &\leq \rho(D) + \epsilon \quad \forall x \in \overline{D}. \end{aligned}$$

and the conclusion follows.

(vi) By (iii) and (v), it is only necessary to prove $\rho(co(D)) \leq \rho(D)$. For any $\alpha > \rho(D)$, there exists $u \in E_+$ such that

$$D \subset [-u, u] + \alpha B_\phi.$$

Because $[-u, u] + \alpha B_\phi$ is a convex set, it follows that

$$co(D) \subset [-u, u] + \alpha B_\phi.$$

So $\rho(co(D)) \leq \alpha$. Let $\alpha \downarrow \rho(D)$ and we get the conclusion. We now give a generalization of the notion of AM-compact mapping studied in [10, Section 123].

Definition 2. Let (E, τ) be a locally convex real linear topological space, which is in addition a vector lattice such that condition (H) is satisfied, and let ϕ be a monotone seminorm in E. Let D be a subset of E and $F : D \to E$ be a τ-continuous mapping, which maps ϕ-bounded sets to ϕ-bounded sets. If F maps each ϕ-almost order bounded subset of D to a τ-relatively compact set, then F is called an *AM-mapping* on D. If for any ϕ-bounded set $S \subset D$, the condition $\rho(S) > 0$ implies $\rho(F(S)) < \rho(S)$, then F is called a *condensing* AM-mapping.

Theorem 3. *Suppose D is a non-empty, ϕ-bounded and τ-closed convex subset in E. If F: $D \to D$ is a condensing AM-mapping, then F has a fixed point in D.*

Proof. Let $x_0 \in D$. Let Z be the collection of all τ-closed convex subsets of D which contain x_0 and are invariant under F. Because $D \in Z$, Z is non-empty. If $S_0 = \bigcap_{S \in Z} S$, then $x_0 \in S_0 \subset D$, S_0 is τ-closed and convex, $F(S_0) \subset S_0$, and $\overline{co}\{F(S_0), x_0\} \subset S_0$ and consequently

$$F(\overline{co}\{F(S_0), x_0\}) \subset F(S_0) \subset \overline{co}\{F(S_0), x_0\}. \tag{1}$$

By (1), $\overline{co}\{F(S_0), x_0\} \in Z$, and from the definition of S_0, we have

$$\overline{co}\{F(S_0), x_0\} = S_0. \tag{2}$$

By Lemma 1, it follows that

$$\rho(S_0) = \rho(\overline{co}\{F(S_0), x_0\}) = \rho(\{F(S_0), x_0\}) = \rho(F(S_0)).$$

As F is a condensing mapping, $\rho(S_0) = 0$, so that S_0 is an almost order -bounded set, and consequently $F(S_0)$ is τ-relatively compact since F is an AM-mapping. From (2), S_0 itself is τ-compact. By the Schauder-Tychonoff fixed-point theorem [6], F has at least one fixed point in $S_0 \subset D$ and the proof is complete.

2. An example.

In this section, we indicate how the approach of the previous section may be applied to give an alternative proof of a recent result of J.Banaś [2] concerning fixed points of the superposition operator on $L^1(0,1)$. Let

$$f(t,x) = f : \ (0,1) \times \mathbf{R}^1 \ \to \mathbf{R}^1$$

satisfy the Carathéodory conditions, that is, f is measurable in t for any x and continuous in x for almost all $t \in (0,1)$. Such a function f will be called a Carathéodory function. If f is a Carathéodory function then the operator F defined (on some appropriate function space on (0,1)) by setting

$$F(x)(t) = f(t, x(t)), \quad t \in (0,1)$$

is known as a Nemytsky or superposition operator. It is a well known result of Krasnosel'skii [8], that F maps $L^1(0,1)$ continuously into itself if and only if there exists a function $a(.) \in L^1(0,1)$ and a non-negative constant b such that

$$|f(t,x)| \leq a(t) + b|x|$$

for all $(t,x) \in (0,1) \times \mathbf{R}^2$. The following result is due to Banaś [2].

Theorem 4. *Let $f : (0,1) \times \mathbf{R}^1 \to \mathbf{R}^1$ be a Carathéodory function and suppose that f satisfies the following conditions*

(i). *f is nondecreasing on $(0,1) \times \mathbf{R}^1$ in the sense that*

$$f(t_1, x_1) \leq f(t_2, x_2)$$

for almost all $(t_1, t_2) \in (0,1)^2$ such that $t_1 \leq t_2$ and for all $x_1 \leq x_2$.

(ii). *There exists a non-negative function $a \in L^1(0,1)$ and a constant b with $0 \leq b < 1$ such that the inequality*

$$|f(t,x)| \leq a(t) + b|x| \tag{3}$$

holds for almost $t \in (0,1)$ *and all* $x \in \mathbf{R}^1$. *If* F *is the Nemytsky operator on* $L^1(0,1)$ *defined by setting*

$$F(x)(t) = f(t, x(t)), \quad t \in (0,1),$$

then F *has a fixed point in* $L^1(0,1)$, *which is non-decreasing in* $(0,1)$.

Proof. We denote by B the unit ball in $L^1(0,1)$. Observe first that if $x, y, z \in L^1(0,1)$ and if

$$x = y + z,$$

then condition (ii) implies that

$$F(x) \in [-v, v] + b\|z\|B,$$

where

$$v = a + b|y|.$$

It follows immediately that F maps almost order-bounded subsets of $L^1(0,1)$ (for the norm topology) to almost order-bounded subsets of $L^1(0,1)$, and if $r = \|a\|(1-b)^{-1}$, then F maps rB to rB. Since the relatively weakly compact subsets of $L^1(0,1)$ are precisely those which are almost order-bounded for norm topology, it follows in particular that F maps almost order-bounded subsets of $L^1(0,1)$ to relatively weakly compact subsets of $L^1(0,1)$. Let Q be the set of all functions in rB that are non-decreasing on $(0,1)$. The set Q is norm closed and convex, hence weakly closed. Since Q is compact in measure and since it is easily seen that F preserves sequential convergence in measure, it follows from a well-known theorem of Vitali [6] that the restriction of F to Q is weakly continuous. It now follows that F is an AM-mapping on Q. To complete the proof, it now suffices, via Theorem 3, to show that F is condensing. We suppose then that $Q_1 \subset Q$ and $\rho(Q_1) > 0$. For any $\epsilon > 0$, there exists a non-negative $u \in L^1(0,1)$, such that

$$Q_1 \subset [-u, u] + (\rho(Q_1) + \epsilon)B.$$

For every $x \in Q_1$, there exists a decomposition

$$x = y + z, \ y \in [-u, u], \ z \in (\rho(Q_1) + \epsilon)B.$$

so that

$$Fx \in [-a - bu, a + bu] + b(\rho(Q_1) + \epsilon)B.$$

It follows that

$$\rho(F(Q_1)) \leq b(\rho(Q_1) + \epsilon).$$

Letting $\epsilon \to 0$, we get the conclusion that F is condensing on Q. By theorem 3, F has a fixed point in Q and the proof is complete.

We remark that the preceding proof, while following the outline of that given in [2] depends in a natural manner on the notion of semi-noncompactness given in section 1 and does not use the expression given in [2] for measures of weak non-compactness in the space $L^1(0,1)$.

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