A Note on Martingales with respect to Complex Measures

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Introduction

Let γ be a rectifiable Jordan curve passing through ∞, and let $z(x)$ denote its arclength parameterization. Assume that γ is a chord-arc curve: this means that there is a constant such that

$$1 \leq \frac{|a-b|}{|\int_a^b z'(x)dx|} = \frac{|a-b|}{|z(a)-z(b)|} \leq C_0 < \infty, \quad \forall a,\ b.$$

Let $\mathcal{D}_k$ denote the ring of sets generated by the collection of dyadic intervals of length 2^{-k}, $k \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers, and define the "conditional expectation" operator E_k by

$$E_k f(t) = \int_I f(x)z'(x)dx \Big/ \int_I z'(x)dx, \quad t \in I,$$

where I is a dyadic interval of length 2^{-k}. The operator E_k has a natural extension to $\mathcal{D}_k$. It may be thought of, in a natural way, as a conditional expectation with respect to the finitely-additive complex measure $z'(x)dx$. In a recent paper, Coifman, Jones and Semmes [CJS] pointed out that this conditional expectation operator has many of the same properties as the conditional expectation with respect to a positive measure. They outlined a proof of the corresponding Littlewood-Paley theorem which made use of a Carleson measure argument, and used the Littlewood-Paley theorem to give a new proof of the L^2-boundedness of Cauchy integrals along chord-arc curves.

In this note we establish a general theory of martingales with respect to complex measures. In our case, the complex measures are defined and σ-additive on a σ-algebra

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of sets $\mathcal{F}$, and satisfy a natural condition with respect to the associated family of sub-σ-algebras $\mathcal{F}_j$. This condition, which generalizes the chord-arc condition for curves, is enough to allow us to prove a number of classical theorems about martingales, but in the complex setting. In particular, we establish, as the main goal of this paper, a Littlewood-Paley theorem. Carleson measure techniques are not available in this context; in their place, we use adaptations of certain methods which can be found, for example, in Garsia's book [G]. To prove the weak type (1, 1) estimate we use a variation of Gundy's lemma.

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1. Conditional expectations with respect to complex measures

Throughout this note we shall work with a fixed complex measure space $(\Omega, \mathcal{F}, d\nu)$ and a sequence of σ-algebras

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_n \subset \ldots \subset \mathcal{F}$$

such that

(i) $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ generates $\mathcal{F}$;

(ii) $\forall n \in \mathbb{Z}^+ = \{1, 2, \ldots n \ \ldots\}, \quad \forall F \in \mathcal{F},$ there exists$\{Uj\} \subset \mathcal{F}_n$ such that

$$F \subset \bigcup U_j.$$

As is well known, there exists a function $\psi \in \mathcal{M}(\mathcal{F})$, the class of the $\mathcal{F}$-measurable functions, and $\psi_n \in \mathcal{M}(\mathcal{F}_n)$ such that

$$|\psi| = |\psi_n| = 1,$$

$$d\nu = \psi |d\nu|, \qquad |d\nu_n| = \psi_n d\nu_n$$

where $d\nu_n = d\nu_{|\mathcal{F}_n}$, and $|d\nu|$ and $|d\nu_n|$ are the total variation measures associated to $d\nu$ and $d\nu_n$, respectively.

By the Radon-Nikodym theorem, there is a function $\mu_n \in \mathcal{M}(\mathcal{F}_n)$ such that $|d\nu_n| = \mu_n |d\nu|_n$. The function μ_n is $\mathcal{F}_n$-measurable and at most 1. We assume throughout the remainder of the paper that the following condition holds: there is a constant C_0 such that, if $\rho_n = \dfrac{1}{\mu_n}$, then

$$||\rho_n||_\infty \leq C_0 < \infty, \qquad \forall n. \tag{1}$$

This condition underlies the definition of conditional expectation, and ensures the validity of the basic results in Lemma 2. Notice that we have the relationship $|d\nu|_n = \rho_n |d\nu_n|$, where ρ_n satisfies (1).

The following lemma guarantees the existence of conditional expectations with respect to complex measures.

Lemma 1. *Assume that condition (1) holds. Let $f \in L^1(|d\nu|)$. Then for every $n \in \mathbb{Z}^+$, there exists an essentially unique function f_n, which is $\mathcal{F}_n$-measurable, such that*

$$\int_A f_n d\nu = \int_A f d\nu$$

for all sets $A \in \mathcal{F}_n$.

Proof. Denote by $\tilde{E}_n$ the conditional expectation operator with respect to the measure $|d\nu|_n$. Then

$$\begin{aligned} \int_A f d\nu &= \int_A f\psi |d\nu| = \int_A \tilde{E}_n(f\psi)|d\nu| \\ &= \int_A \tilde{E}_n(f\psi)|d\nu|_n = \int_A \rho_n \tilde{E}_n(f\psi)|d\nu_n| \\ &= \int_A \psi_n \rho_n \tilde{E}_n(f\psi) d\nu_n = \int_A \psi_n \rho_n \tilde{E}_n(f\psi) d\nu. \end{aligned}$$

Let

$$f_n = \psi_n \rho_n \tilde{E}_n(\psi f),$$

which is a function in $\mathcal{M}(\mathcal{F}_n)$. It is routine to check that f_n is essentially unique modulo the space of null, $\mathcal{F}_n$-measurable functions. □

Definition 1. (Conditional Expectation) The function f_n in Lemma 1 is called *the conditional expectation of f relative to $\mathcal{F}_n$*, and is denoted by $E_n f$ or $E(f|\mathcal{F}_n)$.

Lemma 2. *The conditional expectation operator E_n has the following basic properties:*

(i) $E_n(f) = \tilde{E}_n(\psi f)/\tilde{E}_n(\psi)$;

(ii) E_n *is linear;*

(iii) $\forall A \in \mathcal{F}_n$

$$\int_A |E_n(f)|\ |d\nu| \leq C_0 \int_A |f|\ |d\nu|$$

where C_0 *is the constant appearing in condition (1);*

(iv) $||E_n(f)||_p \leq C_0||f||_p, \quad 1 \leq p \leq \infty$;

(v) if $f \in L^1(|d\nu|)$, $g \in \mathcal{M}(\mathcal{F}_n)$, and $gf \in L^1(|d\nu|)$, *then* $E_n(gf) = gE_n(f)$;

(vi) $E_n(1) = 1$;

(vii) $m \leq n$ *implies* $E_m(E_n f) = E_m f$.

Proof. (i) By the calculation in Lemma 1 we need only verify that

$$\psi_n \rho_n = \frac{1}{\tilde{E}_n(\psi)}.$$

In fact, for $A \in \mathcal{F}_n$,

$$\begin{aligned}\int_A \psi|d\nu| = \int_A d\nu = \int_A d\nu_n = \int_A \frac{1}{\psi_n}|d\nu_n| \\ = \int_A \frac{1}{\psi_n\rho_n}|d\nu|_n = \int_A \frac{1}{\psi_n\rho_n}|d\nu|.\end{aligned}$$

Therefore

$$\tilde{E}_n(\psi) = \frac{1}{\psi_n\rho_n}.$$

(ii) This is a consequence of (i).

(iii) For $A \in \mathcal{F}_n$,

$$\begin{aligned}\int_A |E_n(f)||d\nu| &= \int_A |\rho_n \psi_n \tilde{E}_n(\psi f)|\ |d\nu| \\ &\leq C_0 \int \chi_A |\tilde{E}_n(\psi f)|\ |d\nu| \\ &= C_0 \int |\tilde{E}_n(\chi_A \psi f)|\ |d\nu| \\ &\leq C_0 \int_A |f|\ |d\nu|,\end{aligned}$$

since the operators $\tilde{E}_n$ are contractions on $L^1(|d\nu|)$.

(iv) If $1 \leq p < \infty$,

$$\begin{aligned}\int |E_n(f)|^p |d\nu| &= \int |\rho_n \psi_n \tilde{E}_n(\psi)|^p |d\nu| \\ &\leq C_0^p \int |\tilde{E}_n(\psi)|^p |d\nu| \\ &\leq C_0^p \int |f|^p |d\nu|,\end{aligned}$$

since the operators $\tilde{E}_n$ are contractions on $L^p(|d\nu|)$, $\quad 1 \leq p < \infty$.

The case $p = \infty$ is also a consequence of (i) and the corresponding property of $\tilde{E}_n$.

(v) If $g \in \mathcal{M}(\mathcal{F}_n)$, then

$$E_n(gf) = \psi_n \rho_n \tilde{E}_n(\psi g f) = g \psi_n \rho_n \tilde{E}_n(\psi f) = g E_n(f).$$

(vi) This is a consequence of (i).

(vii) Let $A \in \mathcal{F}_m \subset \mathcal{F}_n$. Then

$$\int_A E_m(f) d\nu = \int_A f d\nu = \int_A E_n(f) d\nu = \int_A E_m(E_n f) d\nu.$$

From the uniqueness we conclude that $E_m f = E_m(E_n f)$. $\square$

Lemma 3. *The following conditions are equivalent.*

1° $||\rho_n||_\infty \leq C_0, \quad \forall n;$

2° $\forall p \in [1,\infty], ||E_n f||_p \leq C_0||f||_p, \forall n, \forall f \in L^p.$

3° $\exists \quad p_0 \in [1,\infty]$ such that $||E_n f||_{p_0} \leq C_0||f||_{p_0}, \quad \forall n, \ \forall f \in L^{p_0}.$

Proof. The proof of Lemma 2 shows that 1° $\Rightarrow$ 2°, while it is obvious that 2° $\Rightarrow$ 3°. We proceed to prove that 3° $\Rightarrow$ 1°. If $p_0 < +\infty$, assumption 3° means that

$$\int |\rho_n|^{p_0}|\tilde{E}_n(\psi f)^{p_0}|d\nu| \leq C_0^{p_0} \int |f|^{p_0}|d\nu|, \quad \forall f \in L^{p_0}.$$

In particular, if $f = \overline{\psi} g, \quad g \in L^{p_0} \cap \mathcal{M}(\mathcal{F}_n)$, we have

$$\int |\rho_n|^{p_0}|g|^{p_0}|d\nu| \leq C_0^{p_0} \int |g|^{p_0}|d\nu|, \quad \forall g \in L^{p_0} \cap \mathcal{M}(\mathcal{F}_n)$$

This implies that

$$||\rho_n||_\infty \leq C_0.$$

If $p_0 = \infty$, replace f by $\overline{\psi} g$, where $g \in L^\infty \cap \mathcal{M}(\mathcal{F}_n)$, in the equality

$$||\rho_n \tilde{E}_n(\psi f)||_\infty \leq C_0||f||_\infty.$$

It follows that

$$||\rho_n g||_\infty \leq C_0||g||_\infty, \quad \forall g \in L^\infty \cap \mathcal{M}(\mathcal{F}_n),$$

and so $||\rho_n||_\infty \leq C_0$. □

Lemma 4. *Let*

$$E^*(f) = \sup_n |E_n(f)|.$$

Then E^* is of strong-type (p,p), $1 < p \leq \infty$, and of weak-type $(1,1)$.

Proof. This is a consequence of the formula

$$E_n(f) = \rho_n \tilde{E}_n(\psi f)$$

and the corresponding result for standard martingales. □

As in the standard case, if a sequence $\{g_n\}_{n=1}^{\infty}$ has the properties $g_n \in \mathcal{M}(\mathcal{F}_n)$ and $E_m(g_n) = g_m, \quad m \leq n$, then we call it a martingale.

2. Littlewood-Paley theory

Denote by L_0^p the space of functions in $L^p(|d\nu|)$ for which $E_0(f) = 0$. If $f \in L_0^1(|d\nu|)$, we define the *square function* of f to be

$$S(f) = \sqrt{\sum_{n=1}^{\infty} |E_n f - E_{n-1} f|^2}.$$

Theorem. *If $1 < p < \infty$, there is a constant C_p such that*

$$||Sf||_p \leq C_p ||f||_p,$$

for all $f \in L_0^p(|d\nu|)$. There is a constant C_1 such that

$$|d\nu|(\{x : Sf > \lambda\}) \leq \frac{C_1}{\lambda} ||f||_1$$

for all $f \in L_0^1(|d\nu|)$.

Remarks on the proof. Among the obstacles to using standard methods to prove the theorem is the fact that E_n is no longer self-adjoint on $L^2(|d\nu|)$; so we do not have orthogonality between the various $(E_n - E_{n-1})$'s. More precisely, the following is no longer true:

$$\int (E_n - E_{n-1}) f \, \overline{(E_m - E_{m-1}) g} \, |d\nu| = 0 \quad (m \neq n).$$

In proving the theorem, we decompose the difference operator $E_n - E_{n-1}$ into two parts: the estimate on the first part reduces to the standard case; the other brings to mind the kind of integral that appears in Carleson measure arguments. We deal with it by using techniques similar to those in Garsia's book [G].

Proof of the case $2 \leq p < \infty$

For $k \in \mathbb{N}$, write

$$S_k(f) = \sqrt{\sum_{n=1}^{k} |E_n f - E_{n-1} f|^2}.$$

Substitute $\alpha = p/2, \quad \rho = (S_k/S_{k-1})^2$ in the following inequality:

$$\rho^\alpha - 1 \leq \alpha(\rho - 1)\rho^{\alpha-1}, \quad \alpha \geq 1, \quad \rho \geq 1.$$

We have

$$\begin{aligned} \int S_n^p(f) &= \sum_{k=1}^{n} \int S_k^p(f) - S_{k-1}^p(f) \\ &\leq \frac{p}{2} \sum_{k=1}^{n} \int S_k^{p-2}(S_k^2 - S_{k-1}^2). \end{aligned}$$

Let

$$\theta_k = S_k^{p-2} - S_{k-1}^{p-2}.$$

We then have that

$$\begin{aligned} \int S_n^p(f) &\leq \frac{p}{2} \sum_{k=1}^{n} \sum_{l=1}^{k} \int \theta_l (S_k^2 - S_{k-1}^2) \\ &= \frac{p}{2} \sum_{l=1}^{n} \sum_{k=l}^{n} \int \theta_l (S_k^2 - S_{k-1}^2) \qquad (2) \\ &= \frac{p}{2} \sum_{l=1}^{n} \int \theta_l (\sum_{k=l}^{n} |\Delta_k f|^2), \end{aligned}$$

where we have written $\Delta_k f = E_k f - E_{k-1} f$. Using the decomposition

$$E_k f - E_{k-1} f = \frac{\tilde{E}_k(\psi f) - \tilde{E}_{k-1}(\psi f)}{\tilde{E}_k(\psi)} - \frac{\tilde{E}_k(\psi) - \tilde{E}_{k-1}(\psi)}{\tilde{E}_k(\psi)\tilde{E}_{k-1}(\psi)} \tilde{E}_{k-1}(\psi f), \qquad (3)$$

we see that the right side of (3) is at most

$$\begin{aligned} C \sum_{l=1}^{n} \int \theta_l \sum_{k=l}^{n} |\tilde{\Delta}_k(\psi f)|^2 + C \sum_{l=1}^{n} \int \theta_l \sum_{k=l}^{n} |\tilde{\Delta}_k \psi|^2 |\tilde{E}_{k-1}(\psi f)|^2 \\ = C I_1 + C I_2 \end{aligned}$$

where we have used the fact that $|\tilde{E}_n(\psi)|^{-1} = |\rho_n| \leq C_0$ a.e., and $\tilde{\Delta}_k g$ denotes $\tilde{E}_k g - \tilde{E}_{k-1} g$.

The estimate of I_1 is standard (see [G, pp. 28–30]):

$$
\begin{aligned}
I_1 &= \sum_{l=1}^{n} \int \theta_l \tilde{E}_l \Big(\sum_{k=l}^{n} |\tilde{\Delta}_k(\psi f)|^2 \Big) \\
&= \sum_{l=1}^{n} \int \theta_l \tilde{E}_l (|\tilde{E}_n(\psi f) - \tilde{E}_{l-1}(\psi f)|^2) \\
&\le 4 \sum_{l=1}^{n} \int \theta_l |\tilde{E}^*(\psi f)|^2 = 4 \int S_n^{p-2} (\tilde{E}^*(\psi f))^2 \\
&\le 4 \Big(\int S_n^p \Big)^{1-\frac{2}{p}} \Big(\int (\tilde{E}^*(\psi f))^p \Big)^{2/p}.
\end{aligned} \tag{4}
$$

To estimate I_2, set

$$
\begin{aligned}
G_n &= \sup_{1 \le k \le n} |\tilde{E}_k(\psi f)|^2, \; G_{-2} = G_{-1} = G_0 = 0 \\
\tau_n &= G_n - G_{n-1}, \tau_0 = \tau_{-1} = 0.
\end{aligned}
$$

Then τ_n is $\mathcal{F}_n$-measurable and $\tau_n \ge 0$. Therefore

$$
\begin{aligned}
I_2 &\le \sum_{l=1}^{n} \int \theta_l \tilde{E}_l \Big[\sum_{k=l}^{n} |\tilde{\Delta}_k \psi|^2 \Big(\sum_{j=l-1}^{k-1} \tau_j + G_{l-2} \Big) \Big] \\
&= \sum_{l=1}^{n} \int \theta_l \tilde{E}_l \Big(\sum_{k=l}^{n} |\tilde{\Delta}_k \psi|^2 \sum_{j=l-1}^{k-1} \tau_j \Big) \\
&\qquad + \sum_{l=1}^{n} \int \theta_l \tilde{E}_l \Big(\sum_{k=l}^{n} |\tilde{\Delta}_k \psi|^2 \cdot G_{l-2} \Big) \\
&= J_1 + J_2,
\end{aligned}
$$

where

$$
\begin{aligned}
J_2 &= \sum_{l=1}^{n} \int \theta_l G_{l-2} \tilde{E}_l \Big(\sum_{k=l}^{n} |\tilde{\Delta} \psi|^2 \Big) \\
&= \sum_{l=1}^{n} \int \theta_l G_{l-2} \tilde{E}_l (|\tilde{E}_n \psi - \tilde{E}_{l-1} \psi|^2) \\
&\le 4 \int \Big(\sum_{l=1}^{n} \theta_l \Big) (\tilde{E}^*(\psi f))^2 \\
&\le 4 \Big(\int S_n^p \Big)^{(p-2)/p} \Big(\int (\tilde{E}^*(\psi f))^p \Big)^{2/p},
\end{aligned} \tag{5}
$$

and

$$J_1 = \sum_{l=1}^{n} \int \theta_l \tilde{E}_l(\sum_{j=l-1}^{n-1} \tau_j \sum_{k=j+1}^{n} |\tilde{\Delta}_k \psi|^2)$$
$$= \sum_{l=1}^{n} \int \theta_l \sum_{j=l-1}^{n-1} \tilde{E}_l(\tau_j \sum_{k=j+1}^{n} |\tilde{\Delta}_k \psi|^2).$$

Since $j+1 \geq l$, we have

$$\begin{aligned}
J_1 &= \sum_{l=1}^{n} \int \theta_l \sum_{j=l-1}^{n-1} \tilde{E}_l(\tilde{E}_{j+1}(\tau_j \sum_{k=j+1}^{n} |\tilde{\Delta}_k \psi|^2) \\
&= \sum_{l=1}^{n} \int \theta_l \sum_{j=l-1}^{n-1} \tilde{E}_l(\tau_j \tilde{E}_{j+1}(\sum_{k=j+1}^{n} |\tilde{\Delta}_k \psi|^2) \\
&= \sum_{l=1}^{n} \int \theta_l \sum_{j=l-1}^{n-1} \tilde{E}_l(\tau_j \tilde{E}_{j+1}(|\tilde{E}_n \psi - \tilde{E}_j \psi|^2) \\
&= \sum_{l=1}^{n} \int \theta_l \sum_{j=l-1}^{n-1} (\tau_j |\tilde{E}_n \psi - \tilde{E}_j \psi|^2) \qquad (6) \\
&\leq 4 \sum_{l=1}^{n} \int \theta_l \sum_{j=l-1}^{n-1} \tau_j \\
&\leq 4 \sum_{l=1}^{n} \int \theta_l \cdot G_{n-1} \\
&\leq 4 \int S_n^{p-2} (\tilde{E}^*(\psi f))^2 \\
&\leq 4(\int S_n^p)^{(p-2)/p} (\int \tilde{E}^*(\psi f)^p)^{2/p}.
\end{aligned}$$

By combining (4), (5) and (6) with the fact that the maximal function operator $\tilde{E}^*$ is bounded on $L^p(|d\nu|)$, we conclude that

$$(\int S_n^p)^{1/p} \leq C_p(\int |f|^p)^{1/p}$$

for some constant C_p independent of f. This finishes the proof for the case $2 \leq p < \infty$.

Proof for the case $1 < p \leq 2$

Since S is a sub-linear operator, it will suffice to show that S is of weak-type (1,1). Then we use the Marcinkiewicz interpolation theorem. We shall use a variant of Gundy's Lemma appropriate to the present context.

Lemma 5. *Let $\lambda > 0$, $f \in L^1(|d\nu|)$. Then there exist $g, H, h, k \in L^1(|d\nu|)$ such that $f = g + H$, $|H| = h + k$ and*

(i) $|d\nu|(\{x : \sup_n |E_n g(x)| > 0\}) \leq \frac{C}{\lambda}||f||_1, \quad ||g||_1 \leq C||f||,$

(ii) $\sum_{n=1}^{\infty} ||\tilde{E}_n h - \tilde{E}_{n-1} h||_1 \leq C||f||_1$, *in particular* $||h||_1 \leq C||f||$,

(iii) $||k||_\infty \leq C\lambda, \quad ||k||_1 \leq C||f||_1.$

Temporarily accepting Lemma 5, let us prove the weak-type (1, 1) inequality for S. In the proof, we use the same letter C to denote constants that may alter from line to line.

By using the sub-linearity of S and the decomposition (3), we have

$$\begin{aligned}
S(f) &\leq S(g) + S(H) \\
&\leq S(g) + C_0 \sqrt{\sum_{n=1}^{\infty} |\tilde{\Delta}_n(\psi H)|^2} + C_0^2 \sqrt{\sum_{n=1}^{\infty} |\tilde{\Delta}_n \psi|^2 \; |\tilde{E}_{n-1}(\psi H)|^2} \\
&\leq S(g) + C_0 S_1 + C_0^2 S_2.
\end{aligned}$$

Now

$$\begin{aligned}
S_2 &\leq \sqrt{\sum_{n=1}^{\infty} |\tilde{\Delta}_n \psi|^2 \; \tilde{E}_{n-1}(|H|)^2} \\
&\leq \sqrt{\sum_{n=1}^{\infty} |\tilde{\Delta}_n \psi|^2 |\tilde{E}_{n-1}(h)|^2} + \sqrt{\sum_{n=1}^{\infty} |\tilde{\Delta}_n \psi|^2 |\tilde{E}_{n-1}(k)|^2} \\
&= T_1 + T_2,
\end{aligned}$$

say.

$$\therefore \quad S(f) \leq S(g) + C_0 S_1 + C_0^2 T_1 + C_0^2 T_2$$

where C_0 is the constant in condition (1). Since $C_0 \geq 1$

$$\{x : S(f) > 4C_0^2\lambda\} \subset \{x : \ S(g) > \lambda\} \cup \{x : S_1 > \lambda\} \cup$$
$$\cup \{x : T_1 > \lambda\} \cup \{x : T_2 > \lambda\}.$$

Now

$$\{x : S(g) > \lambda\} \subset \{x : \ \sup_n |E_n g(x)| > 0\}.$$

So, by Lemma 5(i),

$$|d\nu|(\{x : S(g) > \lambda\}) \leq \frac{C}{\lambda}||f||_1.$$

Since S_1 is a standard square function associated to the standard martingale $\tilde{E}_n(\psi H)$, we have

$$|d\nu|(\{x : S_1 > \lambda\}) \leq \frac{C}{\lambda}||\psi H||_1 \leq \frac{C}{\lambda}||f||_1.$$

To handle T_2, refer to the estimate of I_2 in the proof of the case $2 \leq p < \infty$. This shows that

$$\int {T_2}^2|d\nu| \leq C \int |\frac{k}{\psi}|^2 \leq C\lambda \int |k| \leq C\lambda||f||_1.$$

On the other hand

$$\int {T_2}^2|d\nu| \geq \lambda^2|d\nu|(\{x : T_2 > \lambda\}),$$

so we get the appropriate weak-type (1, 1) estimate for T_2.

Now look at T_1. Notice that

$$\{\sum_{k=1}^{n} \tilde{\Delta}_k\psi.\tilde{E}_{k-1}(h)\}_{n=1}^{\infty}$$

is a martingale in the standard sense, and T_1 is just the corresponding Littlewood-Paley S-function. Therefore, by the standard weak-type (1, 1) inequality ([G, p. 58])

$$|d\nu|(\{x : T_1 > \lambda\}) \leq \frac{C}{\lambda} \int \sup_n |\sum_{k=1}^{n} \tilde{\Delta}_k\psi.\tilde{E}_{k-1}(h)|$$

$$
\begin{aligned}
&= \frac{C}{\lambda}\int \sup_n |\sum_{k=1}^{n} \tilde{\Delta}_k\psi . \sum_{l=1}^{k-1} \tilde{\Delta}_l h| \\
&= \frac{C}{\lambda}\int \sup_n |\sum_{l=1}^{n-1} \tilde{\Delta}_l h \sum_{k=l+1}^{n} \tilde{\Delta}_k\psi| \\
&= \frac{C}{\lambda}\int \sup_n |\sum_{l=1}^{n-1} \tilde{\Delta}_l h(\tilde{E}_n\psi - \tilde{E}_l\psi)| \\
&\leq \frac{C}{\lambda}\int \sup_n \sum_{l=1}^{n-1} |\tilde{\Delta}_l h| \\
&\leq \frac{C}{\lambda}\int \sum_{l=1}^{\infty} |\tilde{E}_l h - \tilde{E}_{l-1} h| \\
&\leq \frac{C}{\lambda}||f||_1
\end{aligned}
$$

by Lemma 5(iii). Now we conclude that

$$|d\nu|(\{x : S(f) > 4C_0\lambda\}) \leq \frac{C}{\lambda}||f||_1.$$

Our last job is to prove the variant of Gundy's Lemma (Lemma 5). We shall use the following concept:

Definition 2. Let $r : \Omega \to \mathbb{Z}^+ \cup \{\infty\}$. Then if $\{x : r(x) = n\} \in \mathcal{F}_n, \forall n$, we call $r(x)$ a *stopping time*. By definition, $\mathcal{F}_\infty = \mathcal{F}$.

Lemma 6. *If $r(x)$ is a stopping time, then*

$$\int_\Omega |f_{r(x)}(x)|\ |d\nu| \leq C_0 \int_\Omega |f(x)|\ |d\nu|$$

where $f_\infty(x) = f(x)$.

Proof.

$$
\begin{aligned}
\int |f_{r(x)}(x)|\ |d\nu| &= \sum_{k=1}^{\infty} \int_{\{x:r(x)=k\}} |f_k(y)|\ |d\nu| + \int_{\{x:r(x)=\infty\}} |f(y)|\ |d\nu| \\
&\leq C_0 \sum_{k=1}^{\infty} \int_{\{x:r(x)=k\}} |f(y)| + \int_{\{x:r(x)=\infty\}} |f(y)| \\
&= C_0 \int |f|\ |d\nu|,
\end{aligned}
$$

by using Lemma 2(iii). □

Lemma 7. *If $r(x)$ is a stopping time, then $f_n^\sharp(x) = f_{n \wedge r(x)}(x)$ is a martingale; in fact we have $f_n^\sharp = E_n(f_{r(x)})$.*

We omit the proof of Lemma 7 since there is no difference from the standard case. The only issue concerns measurability. (See, for example, [L]).

Lemma 8. *If $f \in L^p(|d\nu|)$, $1 \leq p < \infty$, then $E_n f \to f$ in $L^p(|d\nu|)$.*

Proof. As in the standard case, $\forall \varepsilon > 0$, there exist $n \in \mathbb{Z}^+$, and g_n such that $g_n \in \mathcal{M}(\mathcal{F}_n)$ and $||f - g_n||_p \leq \varepsilon$ (for details, see [EG, Chapter 5] for example). Then

$$E_m f - f = E_m(f - g_n) + (E_m g_n - g_n) - (f - g_n).$$

Since $||E_m(f - g_n)||_p \leq C_0 ||f - g_n|| \leq C_0 \varepsilon$, $\forall m$, and if $m > n$, $E_m g_n - g_n = 0$, then

$$\limsup_{m \to +\infty} ||E_m f - f||_p \leq \limsup_{m \to +\infty} ||E_m(f - g_n)||_p + ||f - g_n||_p \leq (a + C_0)\varepsilon.$$

This establishes the desired convergence. □

Now we are in a position to prove Lemma 5.

Proof of Lemma 5. Define $r(x) = \inf\{n : |f_n(x)| > \lambda\}$, with the convention that the infimum of the empty set is taken to be ∞. It is a stopping time, since

$$\{x : r(x) = n\} = \{x : |f_1(x)|, \ldots, |f_{n-1}(x)| \leq \lambda, |f_n(x)| > \lambda\} \in \mathcal{F}_n.$$

Next write $|f_n(x)| = \sum_{k=1}^n \phi_k(x)$, where $\phi_k = |f_k| - |f_{k-1}|$, $\quad f_0 = 0$. Set

$$\varepsilon_n(x) = \phi_n(x) \chi_{\{y : r(y) = n\}}(x).$$

Obviously $\varepsilon_n \geq 0$. Define a new stopping time s by

$$s(x) = \inf\{n : \sum_{k=0}^n \tilde{E}_k(\varepsilon_{k+1})(x) > \lambda\};$$

like $r(x)$, it too is a stopping time.

Now set $t(x) = r(x) \wedge s(x)$. We wish to prove that

$$|d\nu|(\{x : t(x) \neq \infty\}) \leq \frac{C}{\lambda}||f||_1.$$

First of all

$$\{x : t(x) \neq \infty\} \subset \{x : r(x) \neq \infty\} \cup \{x : s(x) \neq \infty\}, \tag{6}$$

and

$$\{x : r(x) \neq \infty\} = \{x : \sup_n |f_n(x)| > \lambda\}$$

$$\therefore \qquad |d\nu|(\{x : r(x) = \infty\}) \leq \frac{C}{\lambda}||f||_1$$

by the maximal martingale Lemma 4. On the other hand

$$\{x : s(x) \neq \infty\} \subset \{x : \sum_{k=0}^{\infty} \tilde{E}_k(\varepsilon_{k+1})(x) > \lambda\}$$

and

$$\int \sum_{k=0}^{\infty} \tilde{E}_k(\varepsilon_{k+1}) = \sum_{k=0}^{\infty} \int \varepsilon_{k+1} = \sum_{k=0}^{\infty} \int_{\{x:r(x)=k+1\}} |f_{k+1}| - |f_k|$$

$$\leq \sum_{k=0}^{\infty} \int_{\{x:r(x)=k+1\}} |f_{k+1}| = \int |f_{r(x)}(x)| \leq C_0||f||_1$$

which gives

$$|d\nu|(\{x : s(x) \neq \infty\}) \leq \frac{C}{\lambda}||f||_1.$$

From the relation (6) we get

$$|d\nu|(\{x : t(x) \neq \infty\}) \leq \frac{C}{\lambda}||f||_1. \tag{7}$$

Let $g(x) = f(x) - f_{t(x)}(x)$, $\quad H(x) = f_{t(x)}(x)$, so that $E_n g = f_n - f_n^\sharp$ where $f_n^\sharp = f_{n\wedge t(x)}(x)$, by Lemma 7, and

$$\{x : \sup_n |E_n g(x)| \neq 0\} \subset \{x : t(x) \neq \infty\}.$$

From (7) it follows that property (i) of Lemma 5 holds. Notice that

$$|f_n^\sharp| = |f_{n\wedge t(x)}(x)| = \sum_{j=1}^{n} (\gamma_j + \varepsilon_j)\chi_{\{y:s(y)\geq j\}},$$

where $\gamma_j = \phi_j \chi_{\{y:r(y)>j\}}$. Set

$$h_n(x) = \sum_{j=1}^{n} (\varepsilon_j - \tilde{E}_{j-1}(\varepsilon_j)) \chi_{\{y:s(y)\geq j\}} = \sum_{j=1}^{n} \psi_j$$

and

$$k_n(x) = \sum_{j=1}^{n} (\gamma_j + \tilde{E}_{j-1}(\varepsilon_j)) \chi_{\{y:s(y)\geq j\}}.$$

Obviously, $h_n + k_n = |f_n^\sharp|$. Since

$$\begin{aligned} \int \sum_{j=1}^{\infty} |\psi_j| &\leq \sum_j \int_{\{y:s(y)\geq j\}} \varepsilon_j + \sum_j \int_{\{y:s(y)\geq j\}} \tilde{E}_{j-1}(\varepsilon_j) \\ &\leq 2 \sum_j \int_{\{y:s(y)\geq j\}} \varepsilon_j \leq 2 \sum_j \int \varepsilon_j \\ &\leq 2 \sum_j \int_{\{x:r(x)=j\}} |f_j| \leq 2C_0 ||f||_1 \end{aligned}$$

from Lemma 5, we conclude that there exists $h \in L^1$ such that $||h||_1 \leq C||f||_1$ and $h_n \to h$ in $L^1(|d\nu|)$. Now from Lemma 8 we also have that $\lim_{n\to\infty} |f_n^\sharp| = \lim_{n\to\infty} |E_n f_{t(x)}| = |f_{t(x)}|$ in $L^1(|d\nu|)$; hence there exists $k \in L^1(|d\nu|)$ such that $||k||_1 \leq C||f||_1$ and $k_n \to k$ in L^1.

It remains to prove that $||k||_\infty \leq C\lambda$. To do this, we shall treat the following two inequalities separately:

(α) $\quad || \sum_{j=1}^{n} \gamma_j \chi_{\{y:s(y)\geq j\}} ||_\infty \leq C\lambda$

(β) $\quad || \sum_{j=1}^{n} \tilde{E}_{j-1}(\varepsilon_j) \chi_{\{y:s(y)\geq j\}} || \leq C\lambda$

As to (α), we have

$$\begin{aligned} \Big| \sum_{j=1}^{n} \gamma_j(x) \chi_{\{y:s(y)\geq j\}}(x) \Big| &= \Big| \sum_{j=1}^{n} \phi_j(x) \chi_{\{y:r(y)>j\}} \chi_{\{y:s(y)\geq j\}}(x) \Big| \\ &= \Big| \sum_{j=1}^{n\wedge r(x)-1\wedge s(x)} \phi_j(x) \Big| \leq \lambda \end{aligned}$$

from the definition of ϕ_j and $r(x)$.

As to (β),

$$
\begin{aligned}
0 &\leq \sum_{j=1}^{n} \tilde{E}_{j-1}(\varepsilon_j) \cdot \chi_{\{y: s(y) \geq j\}} \\
&\leq \sum_{j=1}^{s(x)} \tilde{E}_{j-1}(\varepsilon_j) \\
&= \sum_{j=0}^{s(x)-1} \tilde{E}_j(\varepsilon_{i+1}) \leq \lambda
\end{aligned}
$$

from the definition of $s(x)$.

This completes the proof of Lemma 5. The proof of the theorem is also complete. □

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