31 Nitsche's Conjecture

Conjecture 31.1 (Nitsche) *Let $A \subset \mathbf{R}^3$ be an embedded complete minimal annulus such that $A \cap P_t$ is a Jordan curve for $t_1 < t < t_2$, where $-\infty \le t_1 < t_2 \le \infty$. Then A must be a catenoid. In particular, $t_1 = -\infty$ and $t_2 = \infty$.*

Nitsche made this conjecture in [62], it is still open. We only know that the conjecture is true under certain extra hypotheses, In this section we will give two such theorems. The first one, Theorem 31.2, is due to Nitsche [62]; the proof given here is essentially Nitsche's proof.

Theorem 31.2 *If each $A \cap P_t$ is a starshaped Jordan curve for $t_1 < t < t_2$, then A is a catenoid.*

Proof. By a translation we may assume that $t_1 < 0$, $t_2 > 0$. Let $0 < a < t_2$ and let $A \cap S(0, a)$ be a compact minimal annulus with Jordan curve boundary. By Lemma 9.1 and Proposition 9.2, its conformal structure is

$$A_{R(a)} = \{z \in \mathbf{C} \,\big|\, 1 \le |z| \le R(a)\},$$

where $R(a) > 1$.

Let $X(a) : A_{R(a)} \to \mathbf{R}^3$ be the conformal embedding. Then we know that the third coordinate $X(a)_3$ must be

$$X(a)_3 = \frac{a}{\log R(a)} \log |z|.$$

Let $0 < a < b < t_2$. The moduli of $A_{R(a)}$ and $A_{R(b)}$ are $R(a)$ and $R(b)$ respectively. Since $A \cap S(0, a) \subset A \cap S(0, b)$, we have $R(a) < R(b)$ and thus $A_{R(a)} \subset A_{R(b)}$. We have $X(b) : A_{R(b)} \to \mathbf{R}^3$ such that

$$X(b)_3 = \frac{b}{\log R(b)} \log |z|.$$

It must be that $X(b)_3|_{A_{R(a)}} = X(a)_3$, thus

$$\frac{b}{\log R(b)} \log R(a) = X(b)_3(R(a)e^{i\theta}) = X(a)_3(R(a)e^{i\theta}) = a,$$

which implies that

$$\frac{b}{\log R(b)} = \frac{a}{\log R(a)}. \tag{31.173}$$

Now let $0 < a_1 < a_2 < \cdots < a_n < \cdots t_2$ and $\lim_{n\to\infty} a_n = t_2$; we have $A_{R(a_1)} \subset \cdots \subset A_{R(a_n)} \subset \cdots$. Let $R = \lim_{n\to\infty} R(a_n) \le \infty$. Then the conformal structure of $\mathrm{Int}(A \cap S(0, t_2))$ is the interior of A_R and the conformal embedding is given by $X : \mathrm{Int}(A_R) \to \mathbf{R}^3$ and $X_3(z) = c \log |z|$, where

$$c = \frac{a_n}{\log R(a_n)} > 0$$

is well defined by (31.173).

Let $g : A_R \to \mathbf{C}$ be the Gauss map of $A \cap S(0, t_2)$. As before, we have $\eta = dz/zg(z)$ and the angle of the outward unit normal of $A \cap P_t$ with the x-axis is given by $\psi(r, \theta) = \Im \log g(z)$, where $z = re^{i\theta}$ such that $t = c \log r$. Thus ψ is a multivalued harmonic function. Since each $A \cap P_t$ is a Jordan curve for $0 \leq t < t_2$, we have

$$\psi(r, \theta + 2\pi) = \psi(r, \theta) + 2\pi,$$

which implies that $\Im \log(g(z)/z)$ is a well defined harmonic function in A_R. Thus $h(z) := \log \frac{g}{z}$ is a well defined holomorphic function, and

$$g(z) = ze^{h(z)}. \tag{31.174}$$

The Laurent expansion of h is

$$h(z) = \sum_{-\infty}^{-1} a_n z^n + \sum_{n=0}^{\infty} a_n z^n = h_1(z) + h_2(z);$$

thus h_1 is holomorphic in $\{|z| > 1\} \cup \{\infty\}$ and h_2 is holomorphic in $|z| < R$.

If $\Im h_i$ is bounded for $i = 1, 2$, then $\Re h_i$ is also bounded, and thus $h = h_1 + h_2$ is bounded. In this case, if $R < \infty$, by the Enneper-Weierstrass representation we know that A is not complete. Thus if the $\Im h_i$ are bounded, then $R = \infty$.

Next we prove that indeed $\Im h_i$ are bounded.

Let $D_t \subset P_t$ be the bounded domain bounded by $A \cap P_t$. Let $\alpha(r, \theta) := (X_1, X_2)(re^{i\theta})$ be a parameter representation of $A \cap P_t$, $0 \leq t < t_2$, where $c \log r = t$. For a point $x_0 \in D_t$, let $l(r, \theta)$ be the ray starting from x_0 and passing through $\alpha(r, \theta)$. Consider the angle $\phi(r, \theta)$ made by $l(r, \theta)$ and the x-axis in P_t. We can make ϕ a continuous function of θ such that $\phi(r, \theta + 2\pi) = \phi(r, \theta) + 2m\pi$, where m is an integer depending both on x_0 and α.

Since $A \cap P_t$ is starshaped, there is an $x_t \in D_t$ such that the $l(r, \theta)$ intersects α only at $\alpha(r, \theta)$. Thus for this x_t, $\phi(r, \theta + 2\pi) = \phi(r, \theta) + 2\pi$, and ϕ is a non-decreasing function of θ.

Recall the angle $\psi(r, \theta) = \Im \log g(r^{i\theta})$. Fix $\psi(r, 0) = \Im \log g(r)$ for $1 < r < R$. Comparing the angles ϕ and ψ, by their definitions we have

$$\phi(r, \theta) \leq \psi(r, \theta) + 2n\pi \leq \phi(r, \theta) + \pi/2, \tag{31.175}$$

where the integer n is decided by

$$\phi(r, 0) \leq \psi(r, 0) + 2n\pi \leq \phi(r, 0) + \pi/2.$$

Now for any θ' and θ'' in $[0, 2\pi)$, we have

$$\phi(r,\theta') - \phi(r,\theta'') - \pi/2 \le \psi(r,\theta') - \psi(r,\theta'') \le \phi(r,\theta') - \phi(r,\theta'') + \pi/2. \qquad (31.176)$$

Since $\psi(r,\theta) = \theta + \Im h(re^{i\theta})$, (31.176) gives that

$$\phi(r,\theta') - \phi(r,\theta'') - 5\pi/2 \le \Im h(r^{i\theta'}) - \Im h(r^{i\theta''}) \le \phi(r,\theta') - \phi(r,\theta'') + 5\pi/2,$$

or

$$|\Im h(r^{i\theta'}) - \Im h(r^{i\theta''})| \le 9\pi/2, \qquad (31.177)$$

since $|\phi(r,\theta') - \phi(r,\theta'')| \le 2\pi$.

Now fix z_0 such that $|z_0| = r_0 \in (1, R)$. Define

$$m_i(r) = \min_{|z|=r} \Im h_i(z), \qquad M_i(r) = \max_{|z|=r} \Im h_i(z), \qquad i = 1,\ 2, \qquad (31.178)$$

and

$$s_i(r) = \min_{|z'|=|z''|=r} (\Im h_i(z') - \Im h_i(z'')) = m_i(r) - M_i(r), \qquad (31.179)$$

$$S_i(r) = \max_{|z'|=|z''|=r} (\Im h_i(z') - \Im h_i(z'')) = M_i(r) - m_i(r) = -s_i(r). \qquad (31.180)$$

From the relation

$$\Im h_1(z'') - \Im h_1(z') = \Im h(z'') - \Im h(z') - [\Im h_2(z'') - \Im h_2(z')]$$

we find, using (31.177) and the maximum principle for harmonic functions ($S_2(r) \le S_2(r_0)$ for $0 < r \le r_0$), that

$$|\Im h_1(z'') - \Im h_1(z')| \le 9\pi/2 + S_2(r_0) \quad \text{for } 1 < |z'| = |z''| = r \le r_0. \qquad (31.181)$$

On $|z| = r \le r_0$ we have, denoting by $\hat{z}$ a point with $|\hat{z}| = r$ and $\Im h_1(\hat{z}) = M_1(r)$,

$$\Im h_1(z) = \Im h_1(z) - \Im h_1(\hat{z}) + \Im h_1(\hat{z}) \ge M_1(r) - [9\pi/2 + S_2(r_0)].$$

By the minimum principle, applied to the harmonic function $\Im h_1$ in $1 < |z| \le r_0$, there must be a point z with $|z| = r$ for which $\Im h_1(z) \le \Im h_1(z_0)$, and consequently

$$M_1(r) \le \Im h_1(z_0) + [9\pi/2 + S_2(r_0)]. \qquad (31.182)$$

This inequality, originally derived for $1 < |z| \le r_0$, holds automatically in $r_0 \le |z| < R$ as well because $M_1(r') \le M_1(r)$ for $r \le r'$. An argument similar to the one leading to (31.182) yields

$$m_1(r) \ge \Im h_1(z_0) - [9\pi/2 + S_2(r_0)] \quad \text{for } 1 < r < R. \qquad (31.183)$$

Applying analogous reasoning to the function $\Im h_2$ we find

$$M_2(r) \le \Im h_2(z_0) + [9\pi/2 + S_1(r_0)], \qquad (31.184)$$

$$m_2(r) \geq \Im h_2(z_0) - [9\pi/2 + S_1(r_0)], \tag{31.185}$$

for $1 < r < R$. These relations show that the harmonic functions $\Im h_i$ are bounded from both sides in $1 < |z| < R$ and thus $R = \infty$.

Similarly, consider $A \cap (t_1, 0)$. By the same argument, its conformal type is also $\{1 \leq |z| < \infty\}$.

Thus we know that the conformal type of A is $S^2 - \{p, q\}$. Without loss of generality, we can assume that it is $\mathbf{C} - \{0\}$. Similar argument shows that the third coordinate function can be written as

$$X^3(z) = c \log |z|,$$

where c is a real constant. Then the same argument shows that $g(z) = ze^{h(z)}$ and h is a bounded holomorphic function on $\mathbf{C} - \{0\}$. Passing to the universal covering $\mathbf{C}$ of $\mathbf{C} - \{0\}$ and using Liouville's theorem, h is a constant function. Then by the Enneper-Weierstrass representation, h must be a real constant. Thus $g(z) = az$, $a > 0$ is a real constant, and A must be a catenoid. The proof is complete. □

One observes that if A has finite total curvature, then $K(A) = 2\pi(\chi(A) - 2) = -4\pi$. Corollary 14.6 then tells us that A must be a catenoid. Since A has two annular ends, it is enough to prove that each end has finite total curvature. By Theorem 23.1, we know that if A is properly embedded and if one end of A is above a catenoid, then that end has finite total curvature. Thus if A is a counter-example to Nitsche's conjecture, either it is not properly embedded or one of its two ends is neither above nor below any standard catenoid type barrier. Given the level sets are Jordan curves, such a surface is very hard to imagine its existence.

The second theorem is due to G. D. Crow [11], which shows that uniformly bounded Gauss curvature implies finite total curvature for complete minimal surfaces of conformal type $S^2 - \{p, q\}$.

Theorem 31.3 *Let $X : M = S^2 - \{p, q\} \hookrightarrow \mathbf{R}^3$ be a minimal immersion satisfying:*

1. *$|K| < C$ (M is of bounded Gauss curvature);*

2. *The immersion is given by $X = (X^1, X^2, X^3)$ and is such that the limits as $z \to p$ and $z \to q$ of $X^3(z)$ exist uniformly as extended real numbers.*

Then M is of finite total curvature. In particular, if M is embedded then M is a catenoid.

Proof. We only need prove that M has finite total curvature.

First by Remark 16.4, $|K|$ bounded implies that the convex hull of $X(M)$ is $\mathbf{R}^3$. Thus $X^3(z) \to \pm\infty$ as $z \to p$ or q and the two limits must be different. So without loss of generality we may assume that $M = \mathbf{C} - \{0\}$ and $X^3(z) = c \log |z|$. The Weierstrass data for X then is g and $\eta = (1/zg(z))dz$ and the Gauss curvature is given by

$$K(z) = -\left(\frac{4|z||g||g'|}{(1 + |g|^2)^2}\right)^2.$$

To prove that M has finite total curvature it is enough to prove that g has no essential singularity at either 0 or ∞. Let $h = g^2$ and $r = |z|$, then K can be written as

$$K = -\left(\frac{2r|h'|}{(1+|h|)^2}\right)^2.$$

Now $|K|$ is bounded implies that

$$\frac{2r|h'|}{(1+|h|)^2} < C.$$

Since

$$\frac{1}{(1+|h|)^2} \leq \frac{1}{1+|h|^2} \leq \frac{2}{(1+|h|)^2},$$

$|K|$ is bounded implies that

$$\frac{|z||h'|}{1+|h|^2} < C.$$

The next lemma shows that if h has an essential singularity at ∞, then h cannot miss any value in $\mathbf{C} \cup \{\infty\}$. But $X^3(z) = c \log |z|$ means that the Gauss map g must miss 0 and ∞ in M, since if $g(z_0) = 0$ or ∞ then $|z| = |z_0|$ would not be a level set. Since $h = g^2$, we know that g does not have an essential singularity at ∞.

If h has an essential singularity at 0, using $\zeta = 1/z$, and observing that

$$\frac{|z||h'(z)|}{1+|h(z)|^2} < C, \quad \forall z \in \mathbf{C} - \{0\} \text{ if and only if } \frac{|z||h'(1/z)|}{1+|h(1/z)|^2} < C, \quad \forall z \in \mathbf{C},$$

Thus by the above argument, h and hence g could not have essential singularity at 0 either. Thus g is a meromorphic function on $\mathbf{C} \cup \{\infty\}$ and hence M has finite total curvature as mentioned in Remark 19.3. □

Lemma 31.4 *Let h be a meromorphic function in a neighbourhood U of ∞, and suppose h has an essential singularity at ∞. Suppose h satisfies the inequality*

$$\limsup_{z \to \infty} \frac{|z||h'(z)|}{1+|h(z)|^2} < \infty.$$

Then h cannot omit any value.

Proof. ([48], pages 7 and 8) Let γ be a simple divergent path in U tending to ∞. Then α is said to be an *asymptotic value* of h at ∞ if $h(z) \to \alpha$ as $z \to \infty$ along γ. Suppose h omits the value α. Then by Iversen's Theorem ([65], page 4), α is an asymptotic value at infinity along a simple divergent path γ. By the above theorem, h is normal in U slit along the path γ. By Theorem 2 of [47], page 53, and the remark that follows, h converges uniformly in $U - \gamma$ toward α, no matter of which way z goes to ∞. This contradicts the hypothesis that h has an essential singularity at $z = \infty$. Hence h cannot omit any value. □

Remark 31.5 Under the conditions of Theorem 31.3, if $A := X(M) \cap P_t$ is a Jordan curve, then X is an embedding, so A must be a catenoid. Moreover, by [51] and [85], if A satisfies the condition of Nitsche's conjecture and the Gauss curvature is bounded, then conformally A is $S^2 - \{p, q\}$. Thus Nitsche's conjecture is true if the Gauss curvature is bounded.