28 The Existence of Minimal Annuli in a Slab

Given two Jordan curves Γ_1, Γ_2 in $\mathbf{R}^3$, does $\Gamma := \Gamma_1 \bigcup \Gamma_2$ bound a minimal annulus? This is called the Douglas-Plateau problem which is a generalisation of the original Plateau problem. If the answer to the Douglas-Plateau problem for a given Γ is yes, then we can ask that how many such minimal annuli are there?

These are very hard and interesting problems. Generally, they are attacked with concepts and techniques, such as those from the geometric measure theory which are quite different from the classical setting as in our notes,

One classical result due to Douglas says that if A_1 and A_2 are the areas of least area minimal disks bounded by Γ_1 and Γ_2 respectively, and

$$\inf\{\mathbf{Area}(S)\} < A_1 + A_2,$$

then there is a minimal annulus bounded by Γ. Here the infimum is taken over all surfaces of annular type bounded by Γ. See [13], or [9].

In many cases the answers to the Douglas-Plateau problem are no. One example is that of two coaxial unit circles C_1 and C_2. If the distance d between their centres is large then $C_1 \cup C_2$ cannot bound a catenoid, and therefore as Shiffman's second theorem (Theorem 29.2) shows, $C_1 \cup C_2$ cannot bound a minimal annulus.

When Γ_1 and Γ_2 are smooth convex planar Jordan curves lying in parallel (but different) planes, the Douglas-Plateau problem has a very satisfactory answer. The combined result of Hoffman and Meeks [28], and Meeks and White [53], says,

Let $\Gamma = \Gamma_1 \bigcup \Gamma_2$. Then there are exactly three cases:

1. *There are exactly two minimal annuli bounded by Γ, one is stable and one is unstable.*

2. *There is a unique minimal annulus A bounded by Γ; it is almost stable in the sense that the first eigenvalue of L_A is zero. This case is not generic.*

3. *There are no minimal annuli bounded by Γ.*

4. *Moreover, if A is a minimal annulus bounded by Γ, then the symmetry group of A is the same as the symmetry group of Γ.*

We are not going to discuss the Douglas-Plateau problem in these notes. Rather, we would like to point out some necessary conditions on Γ if it bounds a minimal annulus.

The next theorem is due to Osserman and Schiffer [70], we follow their proof.

Theorem 28.1 *Let δ_1, δ_2, c, d be positive numbers satisfying*

$$\left(\frac{c^2}{2} + d^2\right)^{1/2} \geq \delta_1 + \delta_2. \qquad (28.130)$$

Let Γ_1 *and* Γ_2 *be closed curves in* $\mathbf{R}^3$. *Let*

$$D_1 := \{x \in \mathbf{R}^3 \,|\, x_1^2 + x_2^2 < \delta_1^2, x_3 = 0\},$$

$$D_2 := \{x \in \mathbf{R}^3 \,|\, (x_1 - c)^2 + x_2^2 < \delta_2^2, x_3 = d\}.$$

Then if $\Gamma_1 \subset D_1$ *and* $\Gamma_2 \subset D_2$, *there does not exist any minimal annulus spanning* Γ_1 *and* Γ_2. *More generally, the same conclusion holds if we replace* D_i *by* D_i', $i = 1, 2$, *where*

$$D_1' := \left\{x \in \mathbf{R}^3 \,\Big|\, \left(x_1 - \frac{c}{d}x_3\right)^2 + x_2^2 \le \delta_1^2, x_3 \le 0\right\},$$

$$D_2' := \left\{x \in \mathbf{R}^3 \,\Big|\, \left(x_1 - \frac{c}{d}x_3\right)^2 + x_2^2 \le \delta_2^2, x_3 \ge d\right\}.$$

Remark 28.2 Note that Γ_1 or Γ_2 need not be Jordan curves. Moreover, the theorem is true for minimal annuli in $\mathbf{R}^n$ where $n \ge 3$, with the same proof, see [70].

Suppose $\Gamma_1 \subset P_0$ and $\Gamma_2 \subset P_d$. Let C_1 and C_2 in P_0 and P_d be the smallest circles which enclose Γ_1 and Γ_2 respectively. Let their radii be δ_1 and δ_2. The vertical distance between the centres of C_1 and C_2 is of course d. Let c be the horizontal distance between the centres of C_1 and C_2. Since we can alway adopt coordinates such that C_1 and C_2 are the boundaries of D_1 and D_2 in Theorem 28.1, we conclude that if Γ_1 and Γ_2 span a minimal annulus then

$$\left(\frac{c^2}{2} + d^2\right)^{1/2} \le \delta_1 + \delta_2. \tag{28.131}$$

In case Γ_1 and Γ_2 are Jordan curves, this is a result of Nitsche, see [63].

To prove Theorem 28.1 we need a lemma.

Lemma 28.3 *Let* u *be harmonic in an annulus* $A := \{r_1 \le |z| \le r_2\}$. *Suppose* $b \ge a$, *and*

$$\liminf_{r \to r_1} u(re^{i\theta}) \le a, \quad \limsup_{r \to r_2} u(re^{i\theta}) \ge b.$$

Then for $r_1 < r < r_2$,

$$\int_0^{2\pi} r\frac{\partial u}{\partial r}(re^{i\theta})d\theta \ge 2\pi\frac{b-a}{\log(r_2/r_1)}.$$

Proof. Given $\epsilon > 0$, let

$$v := u - a - \frac{b - a - \epsilon}{\log(r_2/r_1)}\log\frac{r}{r_1}.$$

Then v is harmonic in A, and

$$\liminf_{r\to r_1} v(re^{i\theta}) \le 0, \qquad \limsup_{r\to r_2} v(re^{i\theta}) \ge \epsilon. \tag{28.132}$$

Choose ϵ', $0<\epsilon'<\epsilon$, such that $Dv \neq 0$ on the level curve $C := \{z \in A \,|\, v(z) = \epsilon'\}$. Then C must consist of one or more analytic Jordan curves. But if any subset C' of C bounds a domain $\Omega \subset A$, the function v would be constant on Ω, hence in the whole A, which contradicts (28.132). Thus C consists of a single curve not homologous to zero. Choose δ such that

$$r_1 < \delta < \min_{z\in C} |z|.$$

Then C is homologous to the circle $|z| = \delta$, and hence

$$\int_C \frac{\partial v}{\partial n}\, ds = \int_{|z|=\delta} \frac{\partial v}{\partial n}\, ds.$$

But $v \ge \epsilon'$ outside C and $v = \epsilon'$ on C. Therefore $\partial v/\partial n \ge 0$ on C, where $\partial/\partial n$ is the exterior normal derivative. Thus

$$\int_0^{2\pi} \frac{\partial v}{\partial r}(\delta e^{i\theta})\delta\, d\theta = \int_C \frac{\partial v}{\partial n}\, ds \ge 0.$$

Using the explicit expression for v, we obtain

$$\int_0^{2\pi} \frac{\partial u}{\partial r}(\delta e^{i\theta})\delta\, d\theta \ge 2\pi \frac{b-a-\epsilon}{\log(r_2/r_1)}.$$

Since u is harmonic, the expression on the left side is independent of δ, hence this inequality holds on every circle $|z| = r$. Since ϵ was arbitrary, the lemma is proved. □

Proof of Theorem 28.1. Suppose $X : A = \{r_1 \le |z| \le r_2\} \hookrightarrow \mathbf{R}^3$ is a minimal annulus such that $X\Big|_{|z|=r_i}$ is a parametrisation of Γ_i, $i = 1, 2$. We shall show that (28.130) cannot hold.

We define a function $u(z)$ in A by

$$u(z) = \left(X_1(z) - \frac{c}{d}X_3(z)\right)^2 + X_2^2(z). \tag{28.133}$$

Using the fact that X_i's are harmonic, one can calculate that

$$\Delta u = 2\left(\left|\phi_1 - \frac{c}{d}\phi_3\right|^2 + |\phi_2|^2\right) = 2\left(\left|\phi_1 - \frac{c}{d}\phi_3\right|^2 + |\phi_1 + \phi_3|^2\right)$$

by (6.19).

We assert next that if b is an arbitrary real number then

$$\min\{|w-b|^2 + |w^2+1|\} = \frac{b^2}{2} + 1, \tag{28.134}$$

where the minimum is taken over all complex numbers w. Namely, setting $w = b + re^{i\theta}$ gives

$$\begin{aligned}
|w-b|^2 + |w^2+1| &= r^2 + |b^2 + 2bre^{i\theta} + r^2e^{2i\theta} + 1| \\
&\geq r^2 + b^2 + 1 + 2br\cos\theta + r^2\cos 2\theta \\
&= b^2 + 1 + 2br\cos\theta + 2r^2\cos^2\theta \qquad (28.135) \\
&= b^2 + 1 + 2r^2\left(\cos\theta + \frac{b}{2r}\right)^2 - \frac{b^2}{2} \geq \frac{b^2}{2} + 1.
\end{aligned}$$

This gives a lower bound which is actually attained when $w = b/2$. This proves (28.134).

Returning to Δu, we therefore have

$$\Delta u = 2|\phi_3|^2\left[\left|\frac{\phi_1}{\phi_3} - \frac{c}{d}\right|^2 + \left|\left(\frac{\phi_1}{\phi_3}\right)^2 + 1\right|\right] \geq \left[\left(\frac{c}{d}\right)^2 + 2\right]|\phi_3|^2.$$

Using the notation

$$t = \log r, \quad U(t) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta,$$

we find, as in the proof of Lemma 25.1, that

$$\frac{d^2U}{dt^2} = \frac{1}{2\pi}\int_0^{2\pi} r^2 \Delta u(re^{i\theta})d\theta \geq \frac{c^2+2d^2}{2\pi d^2}\int_0^{2\pi}|\psi_3(re^{i\theta})|^2 d\theta. \qquad (28.136)$$

But

$$\begin{aligned}
2\pi\int_0^{2\pi}|\psi_3(re^{i\theta})|^2 d\theta &\geq \left(\int_0^{2\pi}|\psi_3(re^{i\theta})|d\theta\right)^2 \geq \left(\int_0^{2\pi}\Re[\psi_3(re^{i\theta})]d\theta\right)^2 \\
&= \left(\int_0^{2\pi} r\frac{\partial X_3(re^{i\theta})}{\partial r}d\theta\right)^2 \qquad (28.137)
\end{aligned}$$

by virtue of (25.114). Now the assumption that $\Gamma_1 \subset D_1'$, $\Gamma_2 \subset D_2'$ implies that $X_3(r_1e^{i\theta}) \leq 0$ and $X_3(r_2e^{i\theta}) \geq d$. By Lemma 28.3, we have

$$\int_0^{2\pi} r\frac{\partial X_3(re^{i\theta})}{\partial r}d\theta \geq 2\pi\frac{d}{T}, \qquad (28.138)$$

where

$$T = \log\frac{r_2}{r_1}. \qquad (28.139)$$

Combining (28.136), (28.137), (28.138) gives

$$\frac{d^2U}{dt^2} \geq \frac{c^2+2d^2}{T^2}. \qquad (28.140)$$

By the definition of D_i', the statement $\Gamma_i \subset D_i'$ implies $u(re^{i\theta}) \leq \delta_i^2$, and hence

$$U(t_i) \leq \delta_i^2, \quad i = 1,\ 2. \tag{28.141}$$

We may assume that $t_1 = \log r_1 = 0$ and $t_2 = \log r_2 = T$. Set

$$B = \frac{c^2}{2} + d^2 \tag{28.142}$$

so that (28.140) becomes

$$\frac{d^2U}{dt^2} \geq \frac{2B}{T^2}, \quad 0 < t < T. \tag{28.143}$$

Define $V(t)$ to be the parabola

$$V(t) = at^2 + bt + \delta_1^2,$$

satisfying

$$\frac{d^2V}{dt^2} = \frac{2B}{T^2}, \quad V(0) = \delta_1^2, \quad V(T) = \delta_2^2. \tag{28.144}$$

It follows from (28.141), (28.143), (28.144) that

$$U(t) \leq V(t), \quad 0 < t < T. \tag{28.145}$$

The conditions (28.144) determine the coefficients a, b of V:

$$a = \frac{B}{T^2}, \quad b = \frac{1}{T}(\delta_2^2 - \delta_1^2 - B). \tag{28.146}$$

Since $a > 0$, $V(t)$ has a minimum at $t = t_0$, where

$$t_0 = -\frac{b}{2a} = T\left(\frac{1}{2} - \frac{\delta_2^2 - \delta_1^2}{2B}\right). \tag{28.147}$$

It follows that

$$t_0 > 0 \Leftrightarrow \delta_2^2 - \delta_1^2 < B,$$
$$t_0 < T \Leftrightarrow \delta_2^2 - \delta_1^2 > -B.$$

Thus

$$0 < t_0 < T \Leftrightarrow |\delta_2^2 - \delta_1^2| < B. \tag{28.148}$$

We consider two cases, according to whether (28.148) does or does not hold. If it does not hold, then

$$B \leq |\delta_2^2 - \delta_1^2| = |\delta_2 - \delta_1||\delta_2 + \delta_1| < |\delta_2 + \delta_1|^2. \tag{28.149}$$

On the other hand, if (28.148) does hold, then, by virtue of (28.145) and the fact that $U(t) > 0$ for all t,

$$V(t_0) \geq U(t_0) > 0.$$

But by (28.146) and (28.147),

$$\begin{aligned} V(t_0) &= -\frac{b^2}{4a} + \delta_1^2 > 0 \\ &\Leftrightarrow\ b^2 < 4a\delta_1^2 \Leftrightarrow (\delta_2^2 - \delta_1^2) - 2B(\delta_2^2 + \delta_1^2) + B^2 < 0 \\ &\Rightarrow\ B < (\delta_2^2 + \delta_1^2) + \sqrt{(\delta_2^2 + \delta_1^2)^2 - (\delta_2^2 - \delta_1^2)^2} = (\delta_2 + \delta_1)^2. \end{aligned}$$

Comparing with (28.149), we see in both cases we must have $B < (\delta_1 + \delta_2)^2$. But going back to the definition (28.142) of B, we see that under the assumption that a spanning surface exists, inequality (28.130) must be violated. This proves the theorem. □