24 Complete Minimal Surfaces of Finite Topology

Based on Corollary 24.5, Hoffman and Meeks made the following conjecture in [31]:

Conjecture 24.1 *Let $X : M \hookrightarrow \mathbf{R}^3$ be a properly embedded complete minimal surface of finite topology with more than one end. Then X has finite total curvature.*

With the help of Theorem 23.1, we can give a clearer picture of properly embedded complete minimal surfaces with more than one end.

Theorem 24.2 *Suppose M is a properly embedded minimal surface in $\mathbf{R}^3$ that has two annular ends, each having infinite total curvature. Then these two ends have representatives E_1 , E_2 satisfying the following:*

1. *There exist disjoint closed halfspaces $\mathbf{H}_1$, $\mathbf{H}_2$ such that $E_1 \subset \mathbf{H}_1$ and $E_2 \subset \mathbf{H}_2$.*
2. *All other annular ends of M are asymptotic to flat planes parallel to $\partial\mathbf{H}_1$.*
3. *M has only a finite number of normal vectors parallel to the normal vector of $\partial\mathbf{H}_1$.*

Proof. Given two properly embedded minimal annuli A_1, A_2 each with compact boundary curve, if $A_1 \cap A_2 = \emptyset$ then there exists a standard barrier between them. This means that there exists a half-catenoid or a plane C such that outside of a sufficiently large ball B the barrier C is disjoint from $A_1 \cup A_2$ and also $C \cup B$ separates $A_1 - B$ from $A_2 - B$. Now consider the two annular ends E_1 and E_2 of M with infinite total curvature; Theorem 23.1 implies that C must be a plane. Since C is disjoint from $E_1 \cup E_2$ outside of some ball, $C \cap (E_1 \cup E_2)$ is compact. Hence, after removing compact subannuli of E_1 and E_2, we may choose E_1 and E_2 to lie in the disjoint halfspaces determined by C. The weak maximum principle at infinity (Remark 15.3) implies that E_i and C stay a bounded distance apart for $i = 1, 2$. Therefore, the distance from C to $E_1 \cup E_2$ is greater than some $\epsilon > 0$. It follows that we can choose closed disjoint halfspaces $\mathbf{H}_1$, $\mathbf{H}_2$ with $E_1 \subset \mathbf{H}_1$ and $E_2 \subset \mathbf{H}_2$. This proves the first statement of the theorem.

Suppose now that E_3 is another annular end of M that is disjoint from E_1 and E_2. Corollary 22.6 says that at least one of E_1, E_2 and E_3 lying between two standard barriers. By Proposition 22.3, an end lies between two standard barriers must have finite total curvature. Hence it is evident that E_3 has finite total curvature and lies between two standard barriers, and hence between E_1 and E_2. If E_3 is a catenoid end, then either E_1 or E_2 lies above a catenoid. By Theorem 23.1, E_1 or E_2 has finite total curvature, contradicting our hypotheses. Hence E_3 is asymptotic to a flat plane P. By the weak maximum principle at infinity the end of this plane P stays a positive distance from both E_1 and E_2. This implies that P intersects both E_1 and E_2 in a compact set and hence E_1 and E_2 have proper subends that are a positive distance from P. Hence we may assume that $E_i \cap P = \emptyset$ for $i = 1, 2$. By Theorem 16.1, the convex hulls of

E_1 and E_2 are either a halfspace or a slab since E_1 and E_2 are not compact. Since $E_i \cap P = \emptyset$ for $i = 1, 2$, P must be parallel to $\partial \mathbf{H}_1$. Since E_3 is an arbitrary annular end different from E_1 and E_2, the second part of the theorem is proved.

The proof of the third part of the theorem is quite long. Since we are not interested in the problem of image of Gauss map, we skip it here. The interested reader can read the article [18]. □

We have some direct corollaries of Theorem 23.1.

Corollary 24.3 *Suppose $X : M \hookrightarrow \mathbf{R}^3$ is a smooth properly immersed minimal surface with smooth compact boundary and having finite topology. A sufficient condition for M to have finite total curvature is that $X(M)$ intersects some catenoid in a compact set. If M is embedded, this is also a necessary condition.*

Proof. If M is embedded, has finite total curvature and compact boundary, then the ends of M have a well-defined tangent plane parallel to a fixed plane P, which we take to be the xy-plane. Furthermore, annular end representatives of M can be chosen to be graphs over P, each of some fixed logarithmic growth in terms of $r = \sqrt{x^2 + y^2}$. Any catenoid C with waist circle $P \cap C$, and whose ends are graphs over P with logarithmic growth greater than the logarithmic growths of all the ends of M, must intersect M in a compact set. This proves the necessary part of the theorem.

Now suppose that C is a catenoid such that $B = C \cap X(M)$ is compact. After removing a regular neighbourhood of $X^{-1}(B)$ from M, we may assume that each component of $X(M)$ is disjoint from C. Since M has finite topology, we may assume that, without loss of generality, M is connected and $X(M) \cap C = \emptyset$. Let W and Y be the closures of the components of $\mathbf{R}^3 - C$ and assume W is the component that contains the symmetry axis of C. Thus either $X(M) \subset W$ or $X(M) \subset Y$. For the first case we apply Theorem 23.1 (in fact every annular end has a representative contained in the intersection of W with a halfspace). For the second case we can use Theorem 21.1. □

Corollary 24.4 *Let $X : M \hookrightarrow \mathbf{R}^3$ be a smooth properly embedded minimal surface with smooth compact boundary and having finite topology. Suppose M has two catenoid ends, each a graph over the xy-plane of opposite signed logarithmic growth. Then M has finite total curvature.*

Proof. In this case we may assume that M has a catenoid end E_+ with positive z-coordinate and an end E_- with negative z-coordinate. Since M is proper, every end of M eventually is contained in the region above E_+, below E_-, or in the region between E_+ and E_-. As in the proof of Corollary 24.3, all of the ends of M must have finite total curvature. Thus M has finite total curvature since M has only a finite number of ends. □

Corollary 24.5 *Suppose M is a properly embedded complete minimal surface in $\mathbf{R}^3$ with at least one catenoid type annular end. Then M can have at most one annular ends that is not conformally diffeomorphic to a punctured disk. In particular, if M has finite topology, then M is conformally equivalent to a closed Riemann surface from which a finite number of points, and zero or one closed disks, have been removed.*

Proof. One of the two possible infinite total curvature ends of M must lie above a catenoid, hence by Theorem 23.1 it has finite total curvature. This shows that there is at most one end which has infinite total curvature. □

Remark 24.6 Recently Meeks and Rosenberg [51] proved that if a properly embedded minimal annulus A with smooth compact boundary is contained in a halfspace $\mathbf{H} \subset \mathbf{R}^3$, say $\mathbf{H} = \{(x, y, z), \mid z > 0\}$, then:

1. $A \cap \{(x, y, z) \mid z = c\}$ is a Jordan curve for $c > 0$.
2. The conformal structure of A is a punctured disk.

Combining the above result of Rosenberg and Meeks and Theorem 24.2, we have:

Theorem 24.7 *If $X : M \hookrightarrow \mathbf{R}^3$ is a proper complete minimal embedding with more than one annular end and M has finite topology, then there is a closed Riemann surface S_k of genus k such that*

$$M \cong S_k - \{p_1, \cdots, p_n\}.$$