22 Standard Barriers and The Annular End Theorem

The study of ends of complete minimal surfaces leads to the Annular End Theorem of Hoffman and Meeks [29], and its corollaries.

Theorem 22.1 (The Annular End Theorem) *If M is a properly embedded minimal surface in $\mathbf{R}^3$, then at most two distinct annular ends of M can have infinite total curvature.*

To prove the Annular End Theorem, we need some preparation. First we introduce the notion of a *standard barrier.*

Definition 22.2 A *standard barrier* in $\mathbf{R}^3$ is one of the following two minimal surfaces with boundary: the complement of a disk in a plane in $\mathbf{R}^3$; a component of the complement of a simple, closed, homotopically nontrivial curve on a catenoid.

We will say that a surface $M \subset \mathbf{R}^3$ *admits a standard barrier* if it is disjoint from some standard barrier. We will use the word "eventually" to mean "outside of some sufficiently large compact set of $\mathbf{R}^3$". Thus, two surfaces $M \subset \mathbf{R}^3$ and $N \subset \mathbf{R}^3$ are "eventually disjoint" if they have compact intersection. It is straightforward to see that M admits a standard barrier if and only if it is eventually disjoint from some standard barrier.

Given a standard barrier S and a ball B large enough to contain ∂S, it is clear that $S - B$ divides $\mathbf{R}^3 - B$ into two components. Two surfaces M, $N \subset \mathbf{R}^3$ will be said to be *separated by a standard barrier* if such an S and B can be found so that M and N eventually lie in different components of $\mathbf{R}^3 - (B \cup S)$.

Two disjoint standard barriers divide the complement of a sufficiently large ball $B \subset \mathbf{R}^3$ into three components, only one of which contains portions of both barriers on its boundary. A surface $M \subset \mathbf{R}^3$ that eventually lies in such a component will be said to *lie between two standard barriers.* After a rotation of $\mathbf{R}^3$, if necessary, the region of $\mathbf{R}^3$ between two standard barriers eventually lies in the complement of any $X_c = \{x_1^2 + x_2^2 = (x_3/c)^2\}$ (in the component that contains $P^0 - \{0\}$) for any $c > 0$, no matter how small. It follows from Theorem 21.1 and Remark 21.4 that:

Proposition 22.3 *If $X : M \hookrightarrow \mathbf{R}^3$ is a properly immersed complete minimal surface of finite topology, with compact boundary ∂M, and eventually lies between two standard barriers, then M must have finite total curvature.*

Our strategy in proving the Annular End Theorem is to trap ends between standard barriers. The next lemma contains the critical technical construction.

Before proving the lemma, we introduce the notion of *linking number.*

Definition 22.4 Let γ be an embedded curve in $\mathbf{R}^3$ such that $\mathbf{R}^3 - \gamma$ is homotopic to $\mathbf{R}^2 - \{0\}$. The first homology group of $\mathbf{R}^3 - \gamma$ is $H_1(\mathbf{R}^3 - \gamma) \cong \mathbf{Z}$. Let $\beta \subset \mathbf{R}^3 - \gamma$ be a closed curve. Then the *linking number* of β with γ is the homology class of $[\beta]$ in $H_1(\mathbf{R}^3 - \gamma)$. This is an integer, denoted by $l(\beta, \gamma)$.

If we use the homology group $H_1(\mathbf{R}^3 - \gamma; \mathbf{Z_2})$, then $l_2(\beta, \gamma) = 0$ or 1.

Intuitively, if β is a Jordan curve, then $l(\beta, \gamma) \neq 0$ means that any disk $D \subset \mathbf{R}^3$, such that $\partial D = \beta$, intersects γ. In the homology group $H_1(\mathbf{R}^3 - \gamma; \mathbf{Z_2})$, if γ is a proper curve $\gamma : \mathbf{R} \to \mathbf{R}^3$, then $l_2(\beta, \gamma) \neq 0$ if and only if there is a disk $D \subset \mathbf{R}^3$ such that $\partial D = \beta$ and D intersects γ an odd number of times.

Lemma 22.5 *Suppose M is a properly embedded, piecewise-smooth surface that is a smooth minimal surface outside of some ball and that has at least two ends. Let $\gamma : \mathbf{R} \to M$ be a proper curve that diverges into two distinct ends of M, depending on whether $t \to +\infty$ or $t \to -\infty$. Then M admits a standard barrier whose boundary has linking number 1 with γ.*

Proof. Let $B \subset \mathbf{R}^3$ be a ball large enough to contain the nonsmooth, nonminimal portion of M, and expand it, if necessary, so that the ends of M in question correspond to distinct components of $M - B$. If one has such a ball, any larger one will have the same property. We may also choose B so that ∂B intersects M transversally. Suppose that M_1 and M_2 are the two components of $M - B$ that contain the unbounded components of $\gamma - B$. Since the proper arc γ intersects ∂M_1 and ∂M_2 an odd number of times, we can choose exactly one component of $\mathbf{R}^3 - M$ whose closure, $\mathcal{N}$, has the following property: the arc γ has odd linking number with any 1-cycle in $\mathrm{Int}(\mathcal{N})$ homologous to ∂M_1 in $\mathcal{N}$. Note that ∂M_1 is not homologous to zero in $\mathcal{N}$.

Let $\Sigma_1 \subset \cdots \subset \Sigma_n \subset \cdots$ be an exhaustion of M_1 by smooth compact subdomains, with $\partial M_1 \subset \partial \Sigma_1$. Let $\tilde{\Sigma}_i$ denote a least-area integral current (roughly speaking, piecewise C^1 minimal surface) in $\mathcal{N}$ with boundary $\partial \Sigma_i$, which is $\mathbf{Z_2}$-homologous to Σ_i (i.e., $[\Sigma_i] = [\tilde{\Sigma}_i]$ in $H_1(\mathcal{N}; \mathbf{Z_2})$). Since $\Sigma_i \cup \tilde{\Sigma}_i$ is a boundary in $\mathcal{N}$, $\tilde{\Sigma}_i$ is orientable. Interior regularity of least-area currents (see, for example, [75]) shows that $\tilde{\Sigma}_i \cap \mathrm{Int}(\mathcal{N})$ is a regular embedded minimal surface. Since $\partial \mathcal{N} - \partial B$ has zero mean curvature, the maximum principle and the extension theorem for minimal surfaces imply that either $\tilde{\Sigma}_i \cap (\mathcal{N} - \partial B)$ is regular and $\tilde{\Sigma}_i \cap M_1 = \partial \Sigma_i$ or $\tilde{\Sigma}_i \subset M_1$. Standard compactness theorems imply that a subsequence of the surfaces $\{\tilde{\Sigma}_i\}$ converges to a least-area orientable surface $\Sigma \subset \mathcal{N}$ with $\partial \Sigma = \partial M_1$. Suppose, for the moment, $\Sigma \cap M = \partial \Sigma$.

The surface $\Sigma - \partial B$ is a stable, properly embedded, orientable minimal surface in $\mathbf{R}^3$ with compact boundary and hence has finite total curvature (see, [57], Theorem 1 as well as [19]). Hence, Σ has a finite number, say n, of ends, and each end is asymptotic to a plane or to a catenoid. Let S_R be the sphere of radius R centred at the origin. For R sufficiently large, by Theorem 12.1, $\Sigma \cap S_R$ consists of n parallel almost-great-circles, each of which is the boundary of one of the annular ends of Σ.

By our choice of $\mathcal{N}$, γ has odd linking number with one of the curves in $\Sigma \cap S_R$, and hence has linking number 1 with one of the annular ends of $\Sigma - B_R$. Call this annular end F.

By the weak maximum principle at infinity (Remark 15.3), since $F \cap M = \emptyset$, $\mathbf{dis}(F, M) > 0$. On the other hand, F is asymptotic to an end, C', of a plane or a catenoid. Hence, C' contains a subend, C, whose boundary is a circle that has linking number 1 with γ. Moreover, C is contained in the interior of $\mathcal{N}$. This proves the lemma in the case $\Sigma \cap M = \partial\Sigma$.

In case $\Sigma \subset M$, the extension theorem implies that $\Sigma = M_1$, which means that $\gamma \cap M_1$ is eventually contained in a catenoid-type end or a flat end, say $C'' \subset \Sigma$. This is, in fact, the easier case and can be treated directly, but we prefer to reduce it to the previous case. We may choose C'' so that it is as close as desired to a standard barrier. Moving C'' a small amount in the direction of its limiting normal produces a minimal surface that is disjoint from M_1 and its boundary has linking number with γ equal to either 0 or 1. Move in the direction that makes the linking number equal to 1. The maximum principle at infinity shows that if C'' is moved a small amount, then it is also disjoint from M. We can now apply the argument in the previous case to complete the proof. □

Corollary 22.6 *Suppose M_1, M_2, and M_3 are three pairwise-disjoint, properly embedded minimal surfaces in $\mathbf{R}^3$, each of which has compact boundary and one end. Then at least one of the surfaces lies between two standard barriers.*

Proof. Choose a ball $B \subset \mathbf{R}^3$ that is big enough to contain $\bigcup_i \partial M_i$. The ball can be chosen to intersect $\bigcup M_i$ transversally. After removal of $B \cap M_i$ from each M_i, we may assume that $\partial M_i \subset \partial B$. The curves $\bigcup_{i=1}^3 \partial M_i$ bound a region S on ∂B with the property that the boundary of at least one component of S touches the boundary of more than one of M_i. We will refer to this component as $\mathcal{S}$ and relabel the M_i, if necessary, so that both $\partial\mathcal{S} \cap \partial M_1$ and $\partial\mathcal{S} \cap \partial M_2$ are nonempty.

Let $M = \mathcal{S} \bigcup_i M_i$. We intend to apply Lemma 22.5 to M. Toward that end, choose a proper curve $\gamma : \mathbf{R} \to M$, with $\gamma(\mathbf{R}) \cap \mathcal{S}$ consisting of a single connected arc in $\mathcal{S}$ from ∂M_1 to ∂M_2. We may assume that γ diverges in M_1 (resp. M_2) as $t \to +\infty$ (resp. $t \to -\infty$). By Lemma 22.5, there exists a standard barrier disjoint from M whose boundary has linking number 1 with γ.

We now expand the ball B to be large enough to contain the boundary of this barrier and discard from each M_i the subset $M_i \cap B$. Similarly, let C_1 be the component of the barrier exterior to B. C_1 is still a barrier for M, and γ has linking number 1 with ∂C_1. Moreover, C_1 divides $\mathbf{R}^3 - B$ into two components. Clearly, M_1 and M_2 are in different components. Without loss of generality, we may assume that M_3 is in the same component as M_1.

The curve ∂C_1 divides ∂B into two disks. Let D be the disk containing $\partial M_1 \cup \partial M_3$. We now repeat the construction in the previous paragraph. This time, let S' be a region of D bounded by $\partial C_1 \cup \partial M_1 \cup \partial M_3$. Since ∂C_1 is almost a great circle on ∂B and M_1 and M_3 are in the same component of $\mathbf{R}^3 - C_1$, ∂M_1 and ∂M_3 are in the same half-sphere bounded by ∂C_1, therefore, S' has a component, say $\mathcal{S}'$, with boundary points on both ∂M_1 and ∂M_3. Let $M' = M_1 \cup M_2 \cup M_3 \cup C_1 \cup \mathcal{S}'$. Choose a proper arc $\gamma' \subset M'$ whose intersection with S' lies in $\mathcal{S}'$ and consists of connected arc from ∂M_1 to ∂M_3. (Note that , since $(\partial B - D) \cap \gamma' = \emptyset$, $\partial D = \partial C_1$, γ' has linking number 0 with ∂C_1.) Lemma 22.5 implies that there exists another standard barrier, C_2, that is disjoint from $M_1 \cup M_2 \cup M_3 \cup C_1 \cup S'$ and whose boundary has linking number 1 with γ'.

Expand B again so that $\partial C_2 \subset B$. It is possible to do this so that ∂B meets M transversally. Note that $C_2 \cap \partial B$ is a single closed curve. Again, we discard from M_1, M_2, M_3, C_1 and C_2 the intersection of those surfaces with B. Therefore, all of these surfaces have their boundaries on ∂B.

The barrier C_2 divides $\mathbf{R}^3 - B$ into two regions, as does the barrier C_1. Since they are disjoint, $C_1 \cup C_2$ divides $\mathbf{R}^3 - B$ into three components. Let T_1 (resp. T_2) be the component of $\mathbf{R}^3 - (C_1 \cup C_2)$ whose boundary contains C_1 but is disjoint from C_2 (resp. contains C_2 but is disjoint from C_1). Let F be the third component, whose boundary contains $C_1 \cup C_2$. Since γ' has linking number 1 with ∂C_2, C_2 must separate M_1 from M_3. But clearly, $M_1 \cup M_3 \subset T_1 \cup F$. Hence, either M_1 or M_3 lies in F. That is, either M_1 or M_3 lies between two standard barriers. □

Remark 22.7 Lemma 22.5 and Corollary 22.6 hold even when the minimal surfaces in question are properly immersed rather than properly embedded. The proofs are essentially the same as the proof of the embedded case. See [50] for these types of arguments.

Proof of Theorem 22.1. If M has two or fewer annular ends, there is nothing to prove. If M has three or more annular ends, we apply Corollary 22.6 to any choice of three annular ends of M. It implies that one of them lies between two standard barriers. But by Proposition 22.3, this end must have finite total curvature. Thus, M can have at most two annular ends of infinite total curvature. □

Corollary 22.8 *Suppose M is a properly embedded complete minimal surface in $\mathbf{R}^3$. Then M can have at most two annular ends that are not conformally diffeomorphic to a punctured disk. In particular, if M has finite topology, then M is conformally equivalent to a closed Riemann surface from which a finite number of points, and zero, one, or two pairwise-disjoint closed disks, have been removed.*

Proof. Since any complete annular end of finite total curvature must be conformally a punctured disk, the conclusion is obvious. □