# 21 The Cone Lemma

Let $X_c$ be the cone in $\mathbf{R}^3$ defined by the equation

$$x_1^2 + x_2^2 = (x_3/c)^2, \quad c \neq 0.$$

The complement of $X_c$ consists of three components, two of which are convex. We label the third region $W_c$ and note that $W_c$ contains $P^0 - \{0\}$, where $P^t = \{x_3 = t\}$ for $t \in \mathbf{R}$. Suppose $M \subset W_c$ is a noncompact, properly immersed minimal annulus with compact boundary.

Note that as $c \to 0$, $X_c - \{0\}$ collapses to a double covering of $P^0 - \{0\}$. Note also that any horizontal plane or vertical catenoid is eventually disjoint from any $X_c$, hence eventually contained in $W_c$, no matter how small $c$ is (by "eventually" we mean "outside of a compact set"). Since any embedded complete minimal annular end of finite total curvature is asymptotic to a plane or a catenoid (a graph with logarithmic growth), it follows that, after suitable rotation, such an end is eventually contained in any $W_c$. By Jorge and Meeks' theorem, Theorem 12.1, it is easy to see that a minimally immersed end of finite total curvature with a horizontal limit tangent plane is also eventually contained in every $X_c$. The Cone Lemma [29] shows that this property implies that the annular end must have finite total curvature if it is proper. Hence after a rotation if necessary, a proper minimal annular end has finite total curvature if and only if it is eventually contained in every $X_c$.

Let $A := \{z \in \mathbf{C} \,|\, 1 \leq |z| < \infty\}$.

**Theorem 21.1 (The Cone Lemma)** *Let $X : A \hookrightarrow \mathbf{R}^3$ be a properly immersed minimal annulus with compact boundary. If $M := X(A)$ is eventually contained in $W_c$ for a sufficiently small $c$, then $X$ has finite total curvature.*

In order to prove the Cone Lemma we need to introduce the concept of *foliation.*

**Definition 21.2** Let $M$ be a $C^\infty$ manifold of dimension 3. A $C^k$, $1 \leq k \leq \infty$, foliation of $M$ is a set of leaves $\{\mathcal{L}_\alpha\}_{\alpha \in \mathcal{A}}$ in $M$ that satisfies the following conditions:

1. $\{\mathcal{L}_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of disjoint 2-submanifolds.

2. $\bigcup_{\alpha \in \mathcal{A}} \mathcal{L}_\alpha = M$.

3. For all points $p \in M$ there exists a neighbourhood $U$ of $M$ and class $C^k$coordinate system $(x_1, x_2, x_3)$ of $U$ such that $\mathcal{L}_\alpha \cap U$ is empty or is the solution of $x_3 =$ constant in $U$.

Before proving Theorem 21.1, we will state a fact about the catenoid. Let $C$ be the unit circle in $P^0$ centred at $(0,0)$. Let $C_h$ be the translate of $C$ in the plane $P^h$. There is an $h_2 > 0$ such that for $0 < h < h_2$ there are two catenoids bounded by $C_{-h}$ and $C_h$; one is stable and the other is unstable. While the $C_{-h_2}$ and $C_{h_2}$ bound only one catenoid. When $h > h_2$, there is no catenoid bounded by $C_{-h}$ and $C_h$. It is known that $h_2 \cong 0.6627435$. For more details, see for example [60], §515.

As we will see later, by Schffman's second theorem, Theorem 29.2, any minimal annulus bounded by $C_{-h}$ and $C_h$ must be a catenoid. Thus there is no minimal annulus bounded by $C_{-h}$ and $C_h$ when $h > h_2$.

We describe a technical result that will be used in the proof of the Cone Lemma. Let $\gamma_t$ be the circle of radius $t$ in $P^0$, centred at the origin. Let $C_t^\epsilon$ be the stable catenoid whose boundary circles consists of the vertical translates of $\gamma_t$ by $(0, 0, \pm\epsilon)$. Let $\Delta_\epsilon$ be the solid torus bounded by subsets of $C_2^\epsilon$, $C_4^\epsilon$, $P^\epsilon$, and $P^{-\epsilon}$. Note that $\partial\Delta_\epsilon$ consists of two planar annuli and the two catenoids $C_2^\epsilon$ and $C_4^\epsilon$. Let $K^0 \subset P^0$ be the annulus bounded by $\gamma_2$ and $\gamma_4$. Let $\delta$ be any smooth Jordan curve in $K^0$ that is homotopic to $\gamma_2$ in $K^0$.

**Proposition 21.3** *For $\epsilon > 0$ sufficiently small, $\Delta_\epsilon$ can be foliated by compact minimal annuli $A_t$, $2 \le t \le 4$, with the following properties:*

1. *$A_t = C_t^\epsilon$, for $t$ sufficiently close to 2 or to 4;*

2. *Each $A_t$ meets $P^0$ orthogonally;*

3. *$A_3$ meets $P^0$ in a smooth Jordan curve that converges to $\delta$, in the $C^0$-norm, as $\epsilon \to 0$.*

4. *By selecting suitable $\delta$, the foliation of $\Delta_\epsilon$ satisfies the following: for any $q \in P^0 - \{0\}$, $|q| > 4$, and any line $l$ through $q$, we may rotate $\Delta_\epsilon$ around the $x_3$-axis so that $\overline{0q}$ intersects $\alpha_3^0 = A_3 \cap P^0$ in a point $p$, where $T_pA_3 \cap P^0$ is a line parallel to $l$.*

This proposition is proved in [29], its proof involves several facts about solutions to the Douglas-Plateau problem. Since we are not going to discuss this interesting problem in this lecture notes, we will skip the proof. Readers who are interested in the Douglas-Plateau problem can refer to [55], [56], [53] and [54].

**Proof of Theorem 21.1.** We begin by normalizing the problem.

Let $C(1)$ be the vertical catenoid with waist-circle of radius 1 and denote by $C$ the compact component of $W_c \cap C(1)$. Choose $c > 0$ small enough so that $C$ is a radial graph and foliate $W_c$ by the leaves $\{t \cdot C\}$, $0 < t < \infty$. For convenience, we write $C_t$ for $t \cdot C$.

**Claim 1.** After a homothetic shrinking of $M$ (but not of the foliation) and a discarding of a compact subset of $M$:

1. $\partial M \subset C_1 = C$;

2. $M \subset \bigcup_{1 \le t < \infty} C_t$;

3. $M \cap C_t$ consists of a single closed immersed curve for $t \geq 1$.

**Proof of Claim 1.** Choose $T_0$ large enough so that $\partial M = X(\partial A)$ lies in the bounded component of $W_c - C_{T_0}$. Without loss of generality, we may assume that $C_{T_0}$ intersects $M$ transversally. Denote by $Z$ the closure of the unbounded component of $W_c - C_{T_0}$; that is, $Z = \bigcup_{t \geq T_0} C_t$. Define $f : Z \to [T_0, \infty)$ to be the function whose level set at $t$ is $C_t$. Since the $C_t$ are minimal surfaces, the maximum principle (Theorem 4.4) implies that $f \circ X|_{X^{-1}(Z)}$ has no interior maxima or minima. Moreover, by Theorem 4.4, the intersection of two minimal surfaces in a neighbourhood of a point of tangency consists of $j$ curves, $j \geq 2$, intersecting at that point in equal angles. This implies that $f \circ X|_{X^{-1}(Z)}$ has only index-1 critical points with multiplicity equal to $j - 1$. Therefore, $f$ may have at most $k - 1$ critical points, where $k$ is the first Betti number of $X^{-1}(Z)$, by elementary Morse theory. Consequently, outside of a compact subset of $X^{-1}(Z)$, $f \circ X|_{X^{-1}(Z)}$ is free of critical points. This means that there exists a $T_1 > 0$ such that for $t \geq T_1$, $C_t \cap M$ consists of a finite number of closed immersed curves. Since $M$ has one end, each $C_t \cap M$, $t > T_1$, must consist of a single closed immersed curve. By a similar argument as in the proof of Lemma 11.9 and Theorem 12.1, this time using the maximum principle for minimal surfaces, each $X^{-1}(C_t)$ is a homotopically non-trivial Jordan curve in $A$ and $A' = \bigcup_{t > T_1} X^{-1}(C_t) \subset A$ is an annulus. Conformally $A' \cong A$ since they are both equivalent to the punctured disk.

Discarding the compact subsurface $M \cap (\bigcup_{t \leq T_1} C_t)$ we get $X(A')$. Now rescaling by a factor of $T_1^{-1}$, we satisfy conditions 1, 2, and 3 by denoting $M = X(A')$.

We will write $A'$ as $A$ for convenience.

Because $M$ is properly immersed and projection from $W_c$ to $P^0 - \{0\}$ is also proper, the projection $\Pi \circ X : A \to P^0 - \{0\}$ is a proper map.

**Claim 2.** The mapping $\Pi$ is a submersion outside of a compact set, provided $c > 0$ is sufficiently small.

Before proving **Claim 2**, we will show that the theorem follows from it. By Theorem 19.2, the Gauss map $N$ of $X$ either takes on all points of $\mathbf{C} \cup \{\infty\}$, except for at most a set of capacity zero, or it has a unique limiting value. In the latter case, $X$ must have finite total curvature as remarked in Remark 19.3. But **Claim 2** implies that, outside of some compact set $B$, the Gauss map of $X : A - X^{-1}(B) \hookrightarrow \mathbf{R}^3$ will not take values in the great circle $S^1 \subset S^2$. Since $X$ is proper, $X^{-1}(B)$ is compact. Taking the connected component $W$ in $A - X^{-1}(B)$ which is connected to $\infty$, we infer that the image $N(W)$ is contained in a hemisphere, so the first case of Theorem 19.2 is precluded. Hence, $X$ has finite total curvature.

**Proof of Claim 2.** In this proof, we will need at several points to restrict the size of $c > 0$. At each point, we will continue to assume that $M \subset W_c$. Let $\Delta := \Delta_\epsilon$ be the foliated annulus from Proposition 21.3. Reduce the size of $c$ so that the $\Delta$ has its top and bottom boundaries disjoint from $W_c$. Let $K$ be the intersection of $W_c$ with the vertical cylinder over the disk of radius 4 in $P^0$. Note $\Delta \cap W_c \subset K$. Shrink $c > 0$ even more if necessary, so that the following is true. If the distance from $q \in K$ to $P^0$

is $\tau$, then the vertical translation of $\Delta$ by $\tau$ has the property that its top and bottom boundaries are disjoint from $W_c$.

Suppose now that $\Pi : M \to P^0$ is not a submersion outside of any compact set. This is equivalent to the statement that the points on $M$ with vertical tangent plane form an unbounded set. In particular, there is a point $p \in M - K$ whose tangent plane is vertical.

If $\hat{p}$ is the projection of $p$ onto $P^0$, we can assume that $|\hat{p}| > 4$. According to Proposition 21.3, we may rotate $M$ about the vertical axis so that the following holds: the line $\overline{0\hat{p}}$ intersects $\alpha_3$ at a point where the tangent line to $\alpha_3$ is parallel to $T_pM \cap P^0$. We perform this rotation of $M$ and shrink $M$ so that $\hat{p}$ actually lies on $\alpha_3$. Since the original $M$ satisfied the conditions of **Calim 1**, and the foliation $\{C_t \,|\, 0 < t < \infty\}$ is rotationally symmetric, it follows easily that the modified $M$ also satisfies condition 3 of **Claim 1**. We also discard $M \cap \bigcup_{t<1} C_t$. We will refer to this modified surface as $M$. It is clear that to prove the claim, it is sufficient to prove it for this modified surface.

Vertically translate $\Delta$ so that $\hat{p}$ coincides with $p$, and label this translated torus $\hat{\Delta}$. Also translate the foliation $A_t$ of $\Delta$ to be a foliation $\hat{A}_t$ of $\hat{\Delta}$. Recall that we have chosen $c > 0$ small enough so that the top and bottom boundaries of $\hat{\Delta}$ are disjoint from $W_c$. Also recall that for $t$ near 2 and 4, the leaves of the foliation of $\hat{\Delta}$ are catenoids. Make $c$ smaller, if necessary, to insure that these catenoids are radial graphs.

We will now extend $\hat{A}_t$ to be a smooth foliation of a region that contains $\bigcup_{4 \le t < \infty} C_t$. Let $\hat{A}_t = \frac{t}{4}\hat{A}_4$, $t \ge 4$, be the homothetic expansion of $\hat{A}_t$. The boundary $\partial \hat{A}_2$ consists of two concentric circles. By making $c > 0$ smaller if necessary, we may insure that $C$ is a subset of a stable catenoid $\hat{C}$, whose boundaries are concentric circles exterior to $W_c$ on the parallel planes that contain $\partial \hat{A}_2$. We interpolate between $\partial \hat{A}_2$ and $\partial \hat{C}$ with a smooth family, each member of which is a pair of circles centred on the vertical axis. The vertical distance between circles in each pair is an increasing function of $t$, $1 \le t \le 2$. Note that each pair of circles bounds a unique stable catenoid. Label that catenoid $\hat{A}_t$, $1 \le t \le 2$. It is evident that this family may be chosen to insure that the resulting foliation $\{\hat{A}_t \,|\, 1 \le t < \infty\}$ is smooth. By construction, $M \subset \bigcup_{1 \le t < \infty} \hat{A}_t$, $\partial M \subset \hat{A}_1$, and $\hat{A}_3$ is tangent to $M$ at $p$.

Let $h$ be the smooth function, defined on the union of the leaves $\hat{A}_t$, whose level set at $t$ is $\hat{A}_t$. Restriction of $h$ to $M$ yields a proper function $h \circ X$ on $A$ that satisfies $h \circ X \ge 1$ and is equal to 1 precisely on $\partial A$. Repeating the argument in the proof of **Claim 1** will show that *all* the critical points of $h$ have index 1, possibly with multiplicity. However, $A$ is an annulus and $(h \circ X)^{-1}(1) = \partial A$, so by elementary Morse theory, it follows that $h \circ X$ can have *no* critical points. But $\hat{A}_3$ is tangent to $M$ at $p = X(q) \in M$, which shows that $q$ is a critical point of $h \circ X$. This contradiction completes the proof of **Claim 2** and also of the theorem. □

**Remark 21.4** Let $X_c$ and $W_c$ be as in the Cone Lemma, and $M$ be a proper, connected, complete minimal surface with compact boundary. Then by Theorem 16.1, $M$ is eventually disjoint from $X_c$ is equivalent to the fact that $M$ is eventually contained in $W_c$. Thus suppose that $M$ is a proper, connected, complete minimal surface with compact boundary and finite topology. If after a rotation if necessary, $M$ is eventually disjoint from $X_c$, for some $c > 0$ sufficiently small, then $M$ has finite total curvature.