19 The Gauss Map of Complete Minimal Surfaces

Let $X : M \hookrightarrow \mathbf{R}^3$ be a complete minimal surface. Let g and η be the Weierstrass data for X. The question in this section is how many points does the set $\mathbf{C} \cup \{\infty\} - g(M)$ have? We will only prove a relatively easy theorem due to Osserman, which will be useful when we discuss the behavior of minimal annuli. At the end of this section we will give an up to date survey of partial results for this problem.

To prove the theorem of Osserman mentioned above we have to introduce the concept of *capacity*. Since we only need describe when a set has capacity zero, we will only define zero capacity sets.

Definition 19.1 Let $D \subset \mathbf{C}$ be a closed set. Then D has *capacity zero* if and only if the function $\log(1+|z|^2)$ has no harmonic majorant in $\mathbf{C} - D$, i.e, there is no harmonic function $h : \mathbf{C} - D \to \mathbf{R}$ such that

$$\log(1+|z|^2) \leq h(z), \quad z \in \mathbf{C} - D.$$

Note that any finite set in $\mathbf{C}$ has capacity zero.

Theorem 19.2 *Let $X : M_r(:= \{1 \leq |z| < r \leq \infty\}) \hookrightarrow \mathbf{R}^3$ be a complete minimal surface. Then either the Gauss map g tends to a single limit as $|z| \to r$, or else in each neighbourhood of $\{|z| = r\}$ g takes on all points of $\mathbf{C} \cup \{\infty\}$ except for at most a set of capacity zero.*

Proof. Let $\eta = f(z)dz$. Suppose now that in some neighbourhood of $\{|z| = r\}$ g omits a set Z of positive capacity. This means that for some $1 \leq r_1 < r$, the function $w = g(z)$ omits Z in the domain $D' := \{r_1 < |z| < r\}$. Hence there exists a harmonic function $h(w)$ defined in $\mathbf{C} - Z \supset g(D')$ such that $\log(1+|w|^2) \leq h(w)$. Since the induced metric by X on M_r is $\Lambda^2 = \frac{1}{4}|f|^2(1+|g|^2)^2$, we have

$$\log \Lambda(z) \leq \log \frac{|f|}{2} + h(g(z)).$$

Since g and f are holomorphic, the right hand side of the above inequality is harmonic. By Lemma 10.5 and Proposition 10.6, $r = \infty$. But then g could not have an essential singularity at infinity by Picard's theorem. Thus g tends to a limit, finite or infinite, as z tends to infinity. $\square$

Remark 19.3 Once we know that $r = \infty$ and g has a limit at infinity, we know that X has finite total curvature. The argument is as follows:

By a rotation if necessary we may assume that g has a pole at ∞. Then $g(z) = z^n h(z)$ where $h(\infty) \neq 0$ and $n > 0$. Since

$$\frac{4|g'|^2}{(1+|g|^2)^2} = O(|z|^{-2n}) \quad \text{at } \infty,$$

$$\int_{M_\infty} K dA = -\int_{M_\infty} \frac{4|g'|^2}{(1+|g|^2)^2} dx\, dy > -\infty.$$

Since the Gauss curvature is invariant under rotation, X has finite total curvature.

Combined with Corollary 10.9 we have

Corollary 19.4 *The Gauss map of a complete minimal surface of finite topology achieves every point in* $\mathbf{C}$ *except a set of zero capacity.*

In 1981, F. Xavier [84] proved that the Gauss map of a complete minimal surface cannot miss more than 6 points of $\mathbf{C} \cup \{\infty\}$.

In 1988, Fujimoto [20] proved that the Gauss map of a complete minimal surface cannot miss more than 4 points of $\mathbf{C} \cup \{\infty\}$.

In 1990, Mo and Osserman [58] proved that the Gauss map of a complete minimal surface of infinite total curvature achieves any point of $\mathbf{C} \cup \{\infty\}$, except at most 4, infinitely many times.

Scherk's first surface is an embedded complete doubly periodic minimal surface. One block of it is given by the Weierstrass data

$$g(z) = z, \quad f(z) = \frac{1}{(1+z)(1-z)(z+i)(z-i)}$$

on $\mathbf{C} - \{\pm 1, \pm i\}$. This block has four vertical straight lines as boundary. Rotating 180° around one of those straight lines we get a basic block S of Scherk's first surface. The whole surface is the parallel translations of S in two perpendicular directions. The Gauss map of Scherk's first surface misses 4 points, ± 1, $\pm i$, and takes any other points infinitely many times.

This example shows that Fujimoto's, and Mo and Osserman's results are the best possible results.

All known examples of surfaces whose Gauss map misses 4 points are surfaces of infinite total curvature. This is no surprise, since in 1964, Osserman [67] proved that if the surface has finite total curvature, then g can miss at most 3 points.

The catenoid has finite total curvature -4π and its Gauss map misses 2 points, 0 and ∞, say. Hence either 2 or 3 is the maximal number of points that may be omitted by the Gauss map of a complete minimal surface of finite total curvature. But there is no known example of a complete minimal surface of finite total curvature whose Gauss map misses 3 points in $\mathbf{C} \cup \{\infty\}$.

In 1987, Weitsman and Xavier [81] proved that if g misses 3 points, then the total curvature is less than or equal to -16π.

In 1993, Fang [15] proved that the total curvature must be less than or equal to -20π.

So far the problem of whether 2 or 3 is the maximal number of points that may be omitted by the Gauss map of a complete minimal surface of finite total curvature is still open.