

## 17 Flux

A simple but very useful consequence of minimal surfaces being conformal harmonic immersions is the flux formula. Let  $M$  be a compact domain on a Riemann surface. According to Stoke's Theorem, for any  $C^2$  function  $f : M \rightarrow \mathbf{R}^n$ ,

$$\int_M \Delta_M f dA = \int_{\partial M} df(\vec{n}) ds, \quad (17.66)$$

where  $dA$  is the element of area on  $M$ ,  $\Delta_M$  is the Laplacian on  $M$ ,  $ds$  is the line element on  $\partial M$ ,  $\vec{n}$  is the outward unit normal vector to  $M$  along  $\partial M$ , and  $df(\vec{n})$  is the directional derivative of  $f$  in the direction  $\vec{n}$ . Applying (17.66) to an isometric immersion  $X : M \hookrightarrow \mathbf{R}^3$ , we have that  $dX(\vec{n})$  is the image in  $\mathbf{R}^3$  of the outward *conormal* (i.e.,  $dX(\vec{n})$  is tangent to  $X(M)$  but normal to  $\partial X(M)$ ); writing  $n^* = dX(\vec{n})$  we have

$$\int_M \Delta_M X dA = \int_{\partial M} n^* ds.$$

If  $X$  is minimal and  $M$  is equipped with the metric induced by  $X$ , then

$$\int_{\partial M} n^* ds = 0. \quad (17.67)$$

In particular, if  $\vec{v}$  is any fixed vector in  $\mathbf{R}^3$

$$\int_{\partial M} n^* \bullet \vec{v} ds = 0. \quad (17.68)$$

The integral in (17.68) can be thought of as the tangential part of the flux through  $X(\partial M)$  of the flow in  $\mathbf{R}^3$  with constant velocity vector  $\vec{v}$ . While (17.67) and (17.68) are quite simple and were undoubtedly known in the 19th century, they and their modifications have only recently come into widespread use in the study of minimal and constant mean curvature surfaces [43], [44].

As a sample application of the flux formula, we consider the catenoid. It was Euler who discovered the catenoid, the first nonplanar example of a minimal surface. He did this by finding the surface of revolution that was a critical point for the area functional. Consider a surface of revolution about the  $z$ -axis with profile curve  $(r(t), t)$  in the  $xz$ -plane. Let  $S$  be the compact portion of the surface that is between  $z = t_1$  and  $z = t_2$ .  $S$  is bounded by two circles of radii  $r(t_1)$  and  $r(t_2)$ , respectively. The conormal of  $S$  at the level set  $z = t$  is

$$\frac{1}{\sqrt{1 + r'(t)^2}} (r'(t) \cos \theta, r'(t) \sin \theta, 1).$$

Then computing the flux in the  $z$ -direction ( $\vec{v} = (0, 0, 1)$ ), we get by (17.68)

$$\int_{S \cap \{z=t_1\}} \frac{1}{\sqrt{1 + r'(t_1)^2}} ds = \int_{S \cap \{z=t_2\}} \frac{1}{\sqrt{1 + r'(t_2)^2}} ds$$

or

$$\frac{2\pi r(t_1)}{\sqrt{1+r'(t_1)^2}} = \frac{2\pi r(t_2)}{\sqrt{1+r'(t_2)^2}}.$$

But  $t_1$  and  $t_2$  are arbitrary, so

$$\frac{r(t)}{\sqrt{1+r'(t)^2}} = C, \quad (17.69)$$

where  $C > 0$  is a constant.

The ordinary differential equation (17.69) is satisfied by the functions

$$r(t) = C \cosh(C^{-1}t + B), \quad (17.70)$$

where  $B$  is a constant. These are all possible solutions to (17.69). These curves 17.70 are catenaries, thus, nonplanar minimal surfaces of revolution are all catenoids. The definition of flux and this application are adapted from [33].

Now let us go back to the general theory of flux. Let  $X : M \hookrightarrow \mathbf{R}^3$  be a minimal surface,  $\Gamma \subset M$  a loop. Under the metric induced by  $X$ , we define the *flux of  $X$  along  $\Gamma$*  as

$$\mathbf{Flux}(\Gamma) = \int_{\Gamma} dX(\vec{n})ds, \quad (17.71)$$

where  $\vec{n}$  is the unit vector orthogonal to the unit vector  $\vec{s}$  tangent to  $\Gamma$  and  $(\vec{n}, \vec{s})$  gives the orientation of  $M$ . The flux is well defined on the homology class of  $[\Gamma]$ . In fact, if  $\gamma \in [\Gamma]$  then  $\gamma \cup \Gamma$  bounds a domain  $\Omega$  and we have

$$0 = \int_{\Omega} \Delta_M X dA = \int_{\Gamma} dX(\vec{n})ds - \int_{\gamma} dX(\vec{n})ds.$$

Remember that

$$X(z) = \Re \int_{z_0}^z (\omega_1, \omega_2, \omega_3),$$

where the  $\omega_i$ 's are holomorphic 1-forms. We define the (maybe multiple-valued) harmonic function

$$Y(z) = \Im \int_{z_0}^z (\omega_1, \omega_2, \omega_3).$$

Then  $dX(\vec{n}) = dY(\vec{s})$  by the Cauchy-Riemann equations. Hence we have

$$\mathbf{Flux}(\Gamma) = \int_{\Gamma} dY(\vec{s})ds = Y(\Gamma(l)) - Y(\Gamma(0)) = \Im \int_{\Gamma} (\omega_1, \omega_2, \omega_3),$$

where  $l$  is the arc-length of  $\Gamma$ . Since  $\Gamma$  is a loop, we know that

$$\Re \int (\omega_1, \omega_2, \omega_3) = 0,$$

hence

$$\mathbf{Flux}(\Gamma) = -i \int_{\Gamma} (\omega_1, \omega_2, \omega_3). \quad (17.72)$$

Recall that a holomorphic 1-form  $\phi$  is exact if and only if for any loop  $\Gamma \subset M$ ,  $\int_{\Gamma} \phi ds = 0$ ; if and only if  $\phi = df$ , where  $f$  is a holomorphic function on  $M$ . By the Enneper-Weierstrass representation

$$\omega_1 = \frac{1}{2}(1 - g^2)\eta, \quad \omega_2 = \frac{i}{2}(1 + g^2)\eta, \quad \omega_3 = g\eta,$$

we get the following proposition.

**Proposition 17.1 ([71])** *The following are equivalent:*

1. For each loop  $\Gamma \subset M$ , the flux of  $X$  along  $\Gamma$  vanishes;
2. The holomorphic 1-forms  $\omega_i$  are exact;
3. The holomorphic 1-forms  $\eta$ ,  $g\eta$ , and  $g^2\eta$  are exact;
4. The conjugate immersion  $X_{\pi/2}$  is globally well defined on  $M$ .

Assume that  $X : M \hookrightarrow \mathbf{R}^3$  is a complete minimal surface of finite total curvature. Let  $D - \{p\} \subset M$  be a punctured disk corresponding to an end. We want to calculate the flux of  $X$  along  $\partial D$ . Let  $g$  and  $\eta$  be the Weierstrass data for  $X$ . We have

$$\begin{aligned} 0 &= 2\Re \int_{\partial D} \omega_1 + i2\Re \int_{\partial D} \omega_2 \\ &= \overline{\int_{\partial D} \eta} - \int_{\partial D} g^2\eta = 2\pi i \overline{\mathbf{res}(p, \eta)} - 2\pi i \mathbf{res}(p, g^2\eta) \\ &= -2\pi i [\overline{\mathbf{res}(p, \eta)} + \mathbf{res}(p, g^2\eta)], \end{aligned} \tag{17.73}$$

where  $\mathbf{res}(p, \omega)$  is the residue of  $\omega$  at  $p$ . Hence we have

$$\begin{aligned} \mathbf{res}(p, \omega_1) &= \frac{1}{2} \mathbf{res}(p, \eta) - \frac{1}{2} \mathbf{res}(p, g^2\eta) = \Re \mathbf{res}(p, \eta), \\ \mathbf{res}(p, \omega_2) &= \frac{i}{2} \mathbf{res}(p, \eta) + \frac{i}{2} \mathbf{res}(p, g^2\eta) = -\Im \mathbf{res}(p, \eta). \end{aligned}$$

Since  $\Re \int_{\partial D} \omega_3 = 0$ ,  $c := \mathbf{res}(p, g\eta)$  is real. Thus

$$\begin{aligned} \mathbf{Flux}(\partial D) &= -i \int_{\partial D} (\omega_1, \omega_2, \omega_3) \\ &= 2\pi (\Re(\mathbf{res}(p, \eta)), -\Im(\mathbf{res}(p, \eta)), \mathbf{res}(p, g\eta)). \end{aligned} \tag{17.74}$$

If we write  $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ , then  $X = (X^1, X^2, X^3) = (X^1 + iX^2, X^3)$ , and the flux around  $\partial D$  then is

$$\mathbf{Flux}(\partial D) = 2\pi (\overline{\mathbf{res}(p, \eta)}, \mathbf{res}(p, g\eta)). \tag{17.75}$$

Recall that for any meromorphic 1-form  $\omega$  on a closed Riemann surface  $S_k$ , the summation of residues of  $\omega$  is zero, i.e., for all poles  $p \in S_k$  of  $\omega$ ,

$$\sum_p \mathbf{res}(p, \omega) = 0.$$

Let  $M = S_k - \{p_1, \dots, p_n\}$ , and  $p_i \in D_i$ . Since  $\eta$  and  $g\eta$  can only have poles at an end, we have

$$\sum_{i=1}^n \mathbf{Flux}(\partial D_i) = \sum_{i=1}^n (\overline{\mathbf{res}(p_i, \eta)}, \mathbf{res}(p_i, g\eta)) = 0. \quad (17.76)$$

This is consistent with 17.67 which comes from Stokes' theorem.

When an end is embedded, there is a nicer formula for the flux, i.e., the flux is a vector in the direction of the limiting normal at that end. Let us derive the formula.

After a rotation in  $\mathbf{R}^3$  if necessary, we may assume that  $g(p) = 0$ . Let  $\zeta$  be a coordinate on  $D$  such that  $\zeta(p) = 0$ . Then  $g(\zeta) = \zeta^k \phi(\zeta)$ , where  $k \geq 1$ ,  $\phi$  is holomorphic on  $D$  and  $\phi(0) \neq 0$ . Then  $z = \zeta \phi^{1/k}(\zeta)$  is a coordinate on a (maybe smaller) disk  $D' \subset D$ . Since  $\partial D'$  is homologous to  $\partial D$ ,  $\mathbf{Flux}(\partial D') = \mathbf{Flux}(\partial D)$ . We may assume that  $D' = D$ . Then  $g(z) = z^k$  on  $D$ . By Lemma 11.3, the 1-form  $\eta = f(z)dz$  has a pole of order at least 2 at  $p$ , so

$$f(z) = \frac{1}{z^m} h(z),$$

$m \geq 2$  and  $h$  is holomorphic and  $h(0) \neq 0$ . The Enneper-Weierstrass representation is given by

$$\omega_1 = \frac{1}{2} (z^{-m} - z^{2k-m}) h(z) dz, \quad \omega_2 = \frac{i}{2} (z^{-m} + z^{2k-m}) h(z) dz, \quad \omega_3 = z^{k-m} h(z) dz.$$

As before, since  $X$  is well defined, we have

$$0 = 2\Re \int_{\partial D} \omega_1 + i2\Re \int_{\partial D} \omega_2 = \overline{\int_{\partial D} z^{-m} h(z) dz} - \int_{\partial D} z^{2k-m} h(z) dz. \quad (17.77)$$

If  $2k \geq m$ , then

$$\overline{\int_{\partial D} z^{-m} h(z) dz} = \int_{\partial D} z^{2k-m} h(z) dz = 0,$$

hence

$$\int_{\partial D} \omega_1 = \int_{\partial D} \omega_2 = 0.$$

In particular, if the end is embedded, then  $m = 2$ . When  $k = 1$ ,

$$\int_{\partial D} \omega_3 = 2\pi i h(0).$$

Since

$$\Re \int_{\partial D} \omega_3 = 0,$$

$h(0) \neq 0$  is real. Hence we have

$$\mathbf{Flux}(\partial D) = (0, 0, 2\pi h(0)) = -2\pi h(0)(0, 0, -1). \quad (17.78)$$

When  $k > 1$ , we have

$$\int_{\partial D} \omega_3 = 0,$$

thus

$$\mathbf{Flux}(\partial D) = (0, 0, 0). \quad (17.79)$$

Recall that these two cases corresponding to that the end is catenoid type or flat type. We have the following lemma.

**Lemma 17.2** *Let  $X : S_k - \{p_1, \dots, p_n\} \hookrightarrow \mathbf{R}^3$  be a complete minimal surface of finite total curvature. Let  $E_i := X : D_i - \{p_i\}$  be an embedded end and  $N_i$  be the limiting unit normal at  $p_i$ . Then there is an  $\alpha_i \in \mathbf{R}$  such that*

$$\mathbf{Flux}(\partial D_i) = \alpha_i N_i. \quad (17.80)$$

Furthermore,  $\alpha_i \neq 0$  if and only if  $E_i$  is a catenoid type end.

**Proof.** Let  $A : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  be the rotation such that  $AN_i = (0, 0, -1)$ . Then

$$AX = \Re \int (\omega_1, \omega_2, \omega_3)$$

gives the rotated surface which has limiting normal  $(0, 0, -1)$  at  $p_i$ . When  $E_i$  is a catenoid type end, by (17.78),

$$\begin{aligned} \mathbf{Flux}(\partial D_i) &= \int_{\partial D_i} dX(\vec{n}) ds = A^{-1} \int_{\partial D_i} d(AX)(\vec{n}) ds \\ &= A^{-1} \int_{\partial D_i} -i(\omega_1, \omega_2, \omega_3) ds = -2\pi h(0) A^{-1}(0, 0, -1) \\ &= -2\pi h(0) N_i = \alpha_i N_i. \end{aligned}$$

When  $E_i$  is a flat end, the proof is similar with  $\alpha_i = 0$ . □

**Remark 17.3** Comparing the  $\alpha_i$  here and the  $\alpha$  (the coefficient of the logarithmic term of  $u$ ) in Theorem 11.8 and its proof, we see that  $\alpha_i = 2\pi\alpha$ .

**Theorem 17.4** *Let  $X : S_k - \{p_1, \dots, p_n\} \hookrightarrow \mathbf{R}^3$  be a complete minimal surface of finite total curvature. Suppose that all ends of  $M$  are embedded. Let  $p_i$ ,  $1 \leq i \leq k \leq n$ , correspond to catenoid type ends and  $N_i$  be the corresponding limiting normals. Then  $\{N_i\}_{1 \leq i \leq k}$  are linearly dependent.*

**Proof.** Let  $D_i$  be pairwise disjoint open disks in  $S_k$  such that  $p_i \in D_i$ ,  $1 \leq i \leq n$ . Then  $M' := M - \bigcup_i^n D_i = S_k - \bigcup_i^n D_i$  is compact. Let  $\vec{n}_i$  be the inward unit normal along  $\partial D_i$  in  $D_i$ . We have

$$\begin{aligned} 0 &= \int_{M'} \Delta_X X dA = \int_{\partial M'} dX(\vec{n}) ds = \sum_{i=1}^n \int_{\partial D_i} dX(\vec{n}_i) ds \\ &= - \sum_{i=1}^n \mathbf{Flux}(\partial D_i) = - \sum_{i=1}^k \alpha_i N_i. \end{aligned}$$

Since  $\alpha_i \neq 0$ ,  $\{N_i\}_{1 \leq i \leq k}$  are linearly dependent. □

**Corollary 17.5** *Let  $X : S_k - \{p_1, \dots, p_n\} \hookrightarrow \mathbf{R}^3$  be a complete minimal surface of finite total curvature. Suppose that all ends of  $M$  are embedded. Then  $M$  has either no catenoid type ends, or has at least two catenoid ends.*

*Furthermore, if  $X$  is an embedding, or has parallel embedded ends, then  $M$  has at least two catenoid type ends.*

**Proof.** Straight forward. The last claim is a corollary of the Halfspace Theorem, Theorem 15.1. □