16 The Convex Hull of a Minimal Surface

Recall that the *convex hull* $H(E)$ of a set $E \subset \mathbf{R}^n$ is defined as

$$H(E) = \bigcap_{E \subset \mathbf{H}} \mathbf{H}$$

where $\mathbf{H}$ is a halfspace in $\mathbf{R}^n$. Of course, if E is not contained in any halfspace, then $H(E) = \mathbf{R}^n = \bigcap_{\emptyset} \mathbf{H}$.

We want to study the convex hull of a minimal surface.

Let M be conformally a bounded plane domain and $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface such that X is continuous on $\overline{M}$. If $\partial M \neq \emptyset$, then a simple application of the maximum principle for harmonic functions shows that $X(M) \subset H(X(\partial M))$, where $H(X(\partial M))$ is the convex hull of $X(\partial M)$.

Exercise : Prove this fact.

Now using the the Halfspace Theorem, we can prove more.

Theorem 16.1 **([32])** *Suppose that $M \subset \mathbf{R}^3$ is a proper, complete, connected minimal surface in $\mathbf{R}^3$, whose boundary ∂M, which may be empty, is a compact set. Then exactly one of the following holds:*

1. *$H(M) = \mathbf{R}^3$;*
2. *$H(M)$ is a halfspace;*
3. *$H(M)$ is a closed slab between two parallel planes;*
4. *$H(M)$ is a plane;*
5. *$H(M)$ is a compact convex set. This case occurs precisely when M is compact.*

Furthermore, ∂M has nonempty intersection with each boundary component of $H(M)$.

Remark 16.2 We note that all of these cases are possible. For 1 and 2, examples are the catenoid and half-catenoid. For 3 we could take any of the examples in theorem 14.8 and consider the portion of these surfaces in the slab $|x_3| \leq 1$. This surface is bounded by two Jordan curves. For 4 we have a plane and 5 is the case for any compact example.

Proof of Theorem 16.1. Suppose now that cases 1, 4 and 5 do not occur. To prove that case 2 or case 3 must occur we need show that if H_1 and H_2 are distinct smallest halfspaces containing M, then $P_1 = \partial H_1$ and $P_2 = \partial H_2$ are parallel planes. Suppose now that P_1 and P_2 are not parallel planes. We shall derive a contradiction.

The interior of M cannot have a point in common with $P_1 \cup P_2$. (If it did then the maximum principle for minimal surface (see Theorem 4.4 and Remark 4.6) implies it would have to lie entirely on one plane or the other, contradicting the assumption that 4 does not hold. Let $C = H_1 \cap H_2$.

After a rotation, if necessary, we may assume that C lies in the halfspace $x_3 \geq 0$, that the boundary of C is a graph over the x_1x_2-plane and that $P_1 \cap P_2$ is the x_1-axis. After (if necessary) a translation of M, parallel to the x_1-axis, ∂M lies in the halfspace $x_1 \leq -1$. (This translation leaves C invariant.) In particular $0 \notin M$, and since M is closed (recall that properness implies that M is closed in $\mathbf{R}^3$), there exists an $s > 0$ such that $M \cap B_s = \emptyset$, where $B_s = \{(x_1, x_2, x_3) \,\big|\, (x_1 - s)^2 + x_2^2 + x_3^2 \leq s^2\}$. Let $\Gamma_s = \partial B_s \cap \partial C$. Since Γ_s has a 1-1 projection onto a convex plane curve (recall that ∂C is a graph over the x_1x_2-plane), by Theorem 4.1 it is the boundary of a compact minimal surface Δ_s that is the graph over a convex set in the x_1x_2-plane. By the convex hull property mentioned in the beginning of this section, $\Delta_s \subset B_s$, so Δ_s is a positive distance from M. Note that $B_s \subset \{x_1 \geq 0\}$ and $\Delta_s \subset C \cap \{x_1 \geq 0\}$.

For $t \in [1, \infty)$ consider the surfaces

$$A_t := \{tp \,|\, p \in \Delta_s\}.$$

We note: that each A_t is a nonnegative graph inside of $C \cap \{(x_1, x_2, x_3) \,|\, x_1 \geq 0\}$; that each A_t is compact; that as $t \to \infty$, A_t converges to $\{(x_1, x_2, x_3) \in C \,|\, x_1 = 0\}$; and that every point in $(C \cap \{(x_1, x_2, x_3) \,|\, x_1 > 0\}) - B_s$ lies on some A_t. Because $A = A_1$ is disjoint from M, it follows from an application of the maximum principle that none of the surfaces A_t can meet M (remember that ∂M is a distance at least 1 from any A_t, so any possible contact must occur at an interior point). However $(B_s \cup \bigcup_{t=1}^{\infty} A_t) \supset C \cap \{(x_1, x_2, x_3) \,\big|\, x_1 > 0\}$. Hence $M \subset H_3 = \{(x_1, x_2, x_3) \,\big|\, x_1 \leq 0\}$.

A similar argument will show that for some large positive integer k, $M \subset H_4 = \{(x_1, x_2, x_3) \,\big|\, x_1 \geq -k\}$. Repeating the entire procedure with H_1 and H_3 replacing H_1 and H_2 will prove that M may also be bounded in the x_3-direction and lie in some halfspace $H_5 = \{(x_1, x_2, x_3) \,\big|\, x_3 \leq N\}$ for N sufficiently large. Therefore $M \subset H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5$ which is a compact, convex set. This contradicts the assumption that 5 does not hold. This contradiction completes the proof of the main part of the theorem.

The fact that ∂M intersets each boundary component of $H(M)$ follows from Proposition 15.4. This completes the proof. $\square$

Exercise : Prove that ∂M intersets each boundary component of $H(M)$.

Remark 16.3 All results in this section are true for minimal surfaces with branch points.

Theorem 16.1 is true for minimal submanifolds in $\mathbf{R}^n$, just replace planes by hyperplanes in the theorem.

Remark 16.4 If $X : M \hookrightarrow \mathbf{R}^3$ is a complete minimal surface of finite total curvature, then we know that X is proper. Then by the Halfspace Theorem, Theorem 15.1, $X(M)$

is not contained in any halfspace, and thus $H(X(M)) = \mathbf{R}^3$. This is a case where we know that $H(X(M)) = \mathbf{R}^3$. Here properness is necessary, as Rosenberg and Toubiana [73] have constructed complete minimal annuli which are contained in a slab.

Another example where $H(X(M)) = \mathbf{R}^3$ is a theorem of F. Xavier [85], which says that if $X : M \hookrightarrow \mathbf{R}^3$ is a complete minimal surface with bounded Gauss curvature (i.e, there is an $a > 0$ such that $K(p) > -a$ for any $p \in M$), then $H(X(M)) = \mathbf{R}^3$.