9 Conformal Types of Riemann Surfaces

We will discuss complete minimal surfaces of finite topological type and their annular ends. We need first consider a little of the conformal type of such surfaces.

All *closed* (compact without boundary) 2-dimensional manifolds are classified topologically by their genus and orientability. For example, the topological classification of closed orientable 2-dimensional manifolds is as follows:

The simplest surface is the sphere S^2. Then we can do "surgery" on S^2; by deleting two disjoint disks on S^2 and gluing the boundary of a cylinder along the two circular boundaries we obtain a torus. We say a torus has genus one and is a sphere plus one handle, while S^2 has genus zero, and is a sphere without handle. Thus we obtain genus k surfaces S_k for all integers $k \geq 0$ by adding k handles to a sphere. These are all possible topological types of closed orientable 2-dimensional manifolds. An important topological invariant is Euler's characteristic $\chi(S_k) = 2(1-k)$. Two closed 2-dimensional manifolds are homeomorphic if and only if they have the same Euler's characteristic. Euler's characteristic can be calculated by Gauss-Bonnet Formula, if we have a Riemannian metric on the manifold.

As we have seen before, any smooth orientable 2-dimensional manifold is diffeomorphic to a Riemann surface. Let M and N be two Riemann surfaces. We say that $f : M \to N$ is holomorphic if for any $p \in M$ there is an isothermal coordinate neighbourhood $U \ni p$ with complex coordinate z and an isothermal coordinate $V \ni f(p)$ with complex coordinate w, such that $w \circ f(z)$ is holomorphic.

In the category of Riemann surfaces, $M \cong N$ (*have equivalent conformal type*) if and only if there is a diffeomorphism $f : M \to N$ such that f and f^{-1} are both holomorphic. Such an f is called a *conformal diffeomorphism.*

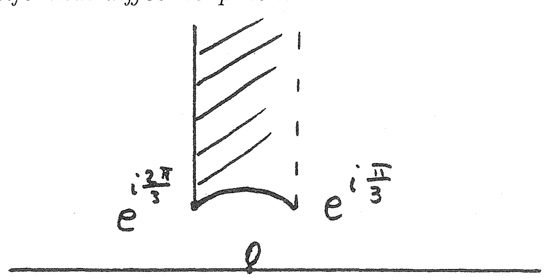

Figure 1

There is considerable interest in classifying the conformal type of closed Riemann surfaces. Although the topological classification is quite simple, the conformal classification is still not clear. In general, S^2 has only one conformal type, i.e., any two *closed* (without boundary) orientable Riemann surfaces of genus zero are conformally diffeomorphic to each other. A typical coordinate system on S^2 is given by stereographic projection from the north and south poles. The conformal structure of genus-one Riemann surfaces corresponds to a region in $\mathbf{C}$ as in the picture above. Such a representation is called a *Riemann moduli space*, hence the Riemann moduli space of the torus is one complex dimension. For genus $k > 1$, it is known that the Riemann moduli space is an algebraic variety of complex dimension $3k - 3$. So far, the Riemann moduli spaces are not well understood.

We are interested in surfaces obtained by making a finite number of punctures and/or removing a finite number of closed disks from a closed Riemann surface M_k of genus k. Suppose that we make n punctures and remove l closed disks, then $M = M_k - (\{p_1, \ldots, p_n\} \cup \bigcup_{i=1}^{l} D_i)$ is said to have *finite topological type*, or just *finite type*, since the topologically (even homotopically) invariant Euler's characteristic is $\chi(M) = 2(1-k) - (n+l)$, which is finite. Thus topologically we cannot tell how many closed disks were removed or how many punctures were made, although we know the sum of them.

When we come to consider the conformal type, removing a closed disk or making a puncture are quite different. For example, making one puncture on S^2 we get the complex plane $\mathbf{C}$, but removing a closed disk on S^2 we get (conformally) the open unit disk $\mathbf{D} := \{z \in \mathbf{C}, \big| |z| < 1\}$. Although $\mathbf{D}$ and $\mathbf{C}$ are C^∞ diffeomorphic to each other, they do have different conformal type. We can see this by Liouville's theorem: were there a conformal diffeomorphism $f : \mathbf{C} \to \mathbf{D}$ then f would be a bounded entire function and hence a constant.

For any connected Riemann surface M without boundary, there is a universal covering Riemann surface $\tilde{M}$ and a holomorphic covering map $f : \tilde{M} \to M$. That f is a covering means for any $p \in M$ there is an open set $U \ni p$ such that $f^{-1}(U)$ consists of disjoint open subsets of $\tilde{M}$ and each component V of $f^{-1}(U)$ is conformally diffeomorphic to U under $f|_V$. Being a universal covering, $\tilde{M}$ must be simply connected. There are only 3 different simply connected Riemann surfaces without boundary, up to conformal diffeomorphism; they are the unit disk $\mathbf{D}$, the unit sphere $\Sigma := S^2 \cong \mathbf{C} \cup \{\infty\}$, and $\mathbf{C}$. It turns out that unless $M = \Sigma$, then $\tilde{M} \neq \Sigma$ (see for example, [1], III, 11G). Hence in general, $\tilde{M}$ is non-compact. An open (non-compact) Riemann surface M is called *parabolic* if there are no non-constant negative subharmonic functions defined on M, otherwise M is *hyperbolic* (see for example, [1], IV, 6). Clearly $\mathbf{D}$ is hyperbolic, since $\Re z - 1$ is a negative subharmonic function. If M is closed then M is called *elliptic*. By maximum principle for subharmonic function, we know that a hyperbolic surface cannot be closed. Thus all Riemann surfaces are divided into three mutually exclusive families.

If $M \subset \mathbf{C}$ is a plane domain with more than one boundary point, then the universal covering is $\mathbf{D}$. In other words, among the planar domains, only $\mathbf{C}$ and $\mathbf{C} - \{p\}$ having $\mathbf{C}$ as universal covering, where $p \in \mathbf{C}$.

An equivalent definition of hyperbolicity of M is that there is a Green's function $G(\zeta, z)$ on M for any $\zeta \in$M which is positive except at ζ and such that in any local coordinate U of ζ, $G(\zeta, z) + \log|z - \zeta|$ is a harmonic function on U. See, for example, [1], IV, 6.

Now return to our $M = M_k - (\{p_1, \ldots, p_n\} \cup \bigcup_{i=1}^{l} D_i)$. At any puncture p or removed closed disk U of M_k, take a larger open disk $D \subset M_k$ such that $p \in D$ or $U \subset D$. Since such (topological) annuli ($D - U$ or $D - \{p\}$) determine the global properties of our complete minimal surfaces we should understand more about their conformal structure. The rest of this section is devoted to the description of the conformal structures of annuli.

Among all doubly connected domains (annuli) in $\mathbf{C}$ we consider the special case that the doubly connected domains D are bounded by two Jordan curves (embedded closed curves), such domains are called *annuli with Jordan curve boundaries.*

Without loss of generality, we can assume that $\partial D = C_1 \cup C_2$, where C_1 and C_2 are analytic Jordan curves. Perhaps the best way to see that this is true is by the following picture, of course we assume the Riemann Mapping Theorem.

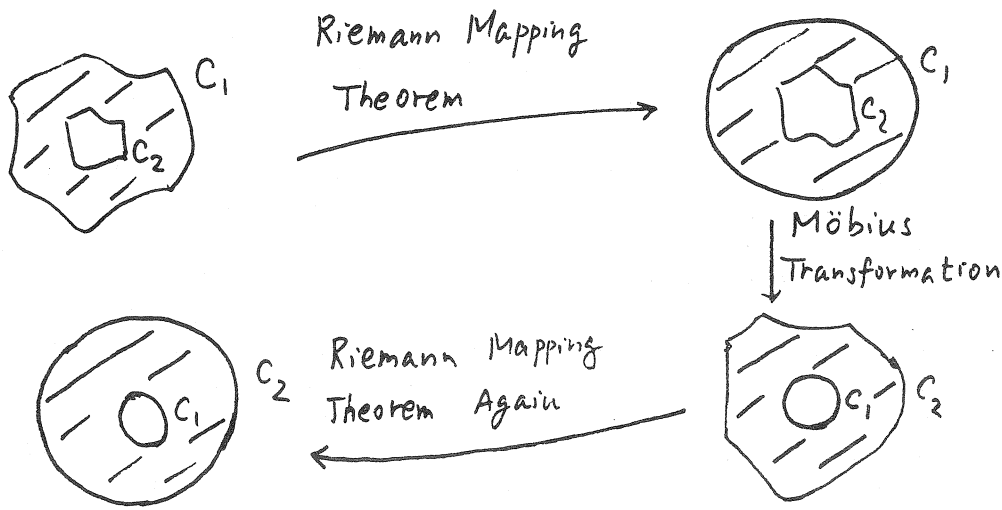

Figure 2

Let $\exp(z) = e^z$. Consider the universal covering $f = \exp : \mathbf{C} \to \mathbf{C} - \{0\} \supset \overline{D}$; then $f^{-1}(\overline{D})$ is a simply connected domain in $\mathbf{C}$ with analytic boundary. $\Re f^{-1}(\overline{D})$ is bounded and for any $w \in f^{-1}(\overline{D})$, $w + 2n\pi i \in f^{-1}(\overline{D})$ for any $n \in \mathbf{Z}$. In fact, $f^{-1}(z) = \log z$, a multivalued holomorphic function. By the Riemann Mapping Theorem, for any $b > 0$ there is a conformal diffeomorphism

$$\phi : f^{-1}(\overline{D}) \to S_b = \{z \in \mathbf{C} : 0 \leq \Re z \leq b\}$$

such that $\phi[f^{-1}(C_1)] =$ the y-axis. Any two such maps ϕ_1 and ϕ_2 induce a conformal diffeomorphism

$$h = \phi_2 \circ \phi_1^{-1} : S_b \to S_b$$

which maps the y-axis and the straight line $\Re z = b$ onto themselves. By Schwarz's reflection principle, h can be extended to a conformal diffeomorphism from $\mathbf{C}$ to $\mathbf{C}$, hence $h(z) = az + e$ since it maps ∞ to ∞. We have $a(i\,y) + e = i\,u$, $a(b + i\,y) + e = b + i\,d$, where y is any real number and u, d are real numbers. It must be that $a = 1$ and e is a pure imaginary number $e = i\,c$. So we have $\phi_2 = \phi_1 + i\,c$. Now take

$$\psi(w) = \phi(w + 2\pi i).$$

Then ψ is a conformal diffeomorphism from $f^{-1}(\overline{D})$ to S_b, so $\psi = \phi + i\,c$ for some number $c \in \mathbf{R}$. If $c \neq 0$, then take

$$\frac{b'}{b}\psi = \frac{b'}{b}\phi + i\,\frac{b'}{b}c,$$

which is a conformal diffeomorphism from $f^{-1}(\overline{D})$ to $S_{b'}$. By adjusting b' we can get $b'c/b = \pm 2\pi$, then

$$\frac{b'}{b}\phi(w + 2\pi i) = \frac{b'}{b}\phi(w) \pm 2\pi i.$$

Denoting $\frac{b'}{b}\phi$ by ϕ, we can define a conformal diffeomorphism from $\overline{D}$ to the annular ring

$$\{z \in \mathbf{C} : 1 \leq |z| \leq e^{b'}\}$$

by $g(z) = e^{\phi(f^{-1}(z))}$. If $c = 0$, we can do this directly. Thus we have proved:

Lemma 9.1 *Each annulus with Jordan curve boundaries is conformally equivalent to an annular ring*

$$\triangle = \{z \in \mathbf{C} : r \leq |z| \leq R\}.$$

Let

$$D = \{z \in \mathbf{C} : \rho \leq |z| \leq P\}$$

be another annular ring which is conformally diffeomorphic to $\triangle$, i.e., there is a holomorphic hemeomophism $h : D \to \triangle$ such that h maps $\{|z| = \rho\}$ to $\{|z| = r\}$ and $\{|z| = P\}$ to $\{|z| = R\}$. By repeatedly using Schwarz's reflection principle, we can extend h to a conformal diffeomorphism $h : \mathbf{C} \to \mathbf{C}$ such that $h(0) = 0$ and $h(\infty) = \infty$. Hence $h(z) = az$. Thus we will have $|a| = r/\rho = R/P$, and

$$M := \frac{R}{r} = \frac{P}{\rho}. \tag{9.37}$$

The number M defined in (9.37) is called *modulus.* For an annulus with Jordan curve boundaries we can define its modulus by Lemma 9.1. We have just proved that if D and $\triangle$ are conformally equivalent, then they have the same moduli. On the other hand, if D and $\triangle$ are annular rings which have the same moduli, then $h(z) = \frac{r}{\rho}z$ is a conformal diffeomorphism from D to $\triangle$. Thus we have:

Proposition 9.2 *Two annuli with Jordan curve boundaries are conformally equivalent if and only if they have the same moduli.*

Thus the interior of any annulus with Jordan curve boundary is conformally equivalent to

$$A_R = \{z \in \mathbf{C} : \frac{1}{R} < |z| < R\}, \tag{9.38}$$

for some $R > 1$.

Letting $R \to \infty$, we get $\mathbf{C} - \{0\}$ which topologically is an annulus. Its universal covering space is $\mathbf{C}$, as we have seen. Therefore it is parabolic, and different from the A_R for $1 < R < \infty$, which are hyperbolic.

The remaining case is the punctured disk $\mathbf{D}^* := \mathbf{D} - \{0\}$ which is conformally equivalent to $\{z \in \mathbf{C}, \Big| \rho \leq |z| < \infty\}$ for any $\rho > 0$. Since $\Re z - 2$ is a non-constant negative harmonic function on $\mathbf{D}^*$, $\mathbf{D}^*$ is hyperbolic. We can naturally think of D^* as having ∞ modulus. This suggests that D^* is different from annuli with Jordan curve boundaries. Later we will see that this is indeed true. Our proof actually uses minimal surfaces, see the next section.