8 Some Applications of the Enneper-Weierstrass Representation

Given a minimal surface $X: M \hookrightarrow \mathbf{R}^3$ and its Enneper-Weierstrass representation, fix a simply connected open set $U \subset M$. Fixing $p_0 \in U$ we can define a family of isometric minimal surfaces associated to $X_\theta : U \to \mathbf{R}^3$, $0 \leq \theta < 2\pi$, by

$$X_\theta(p) = \Re e^{i\theta} \int_{p_0}^{p} (\omega_1, \omega_2, \omega_3) + C, \quad p \in U, \tag{8.35}$$

where the ω_i's are the 1-forms in the Enneper-Weierstrass representation of X and C is a constant vector. The Enneper-Weierstrass data for X_θ are $g_\theta = g$ and $\eta_\theta = e^{i\theta}\eta$. When $\theta = \pi/2$, $X_{\pi/2}$ is called the *conjugate surface* of X.

Let $I \subset \mathbf{R}$ be an interval and $r : I \to U$ be a geodesic such that $X \circ r$ is a plane curve. Then we know that r must be a curvature line, thus by our criterion in the previous section,

$$d(g \circ r)\eta \circ r \in \mathbf{R}.$$

Since X and $X_{\pi/2}$ are isometric, r is also a geodesic of $X_{\pi/2}$. Moreover,

$$d(g_{\pi/2} \circ r)\eta_{\pi/2} \circ r = id(g \circ r)\eta \circ r \in i\mathbf{R},$$

and hence r is an asymptotic line of $X_{\pi/2}$. Since the space curve $X_{\pi/2} \circ r$ is both a geodesic and an asymptotic line of $X_{\pi/2}$, it must be a straight line segment on $X_{\pi/2}$ (in fact, the normal and geodesic curvatures of $X_{\pi/2} \circ r$ are both zero, and so its curvature is zero everywhere). Since X and $X_{\pi/2}$ are conjugate to each other (up to sign), we have

Proposition 8.1 *$X \circ r$ is a plane geodesic (straight line segment) if and only if $X_{\pi/2} \circ r$ is a straight line segment (a plane geodesic).*

In fact, we have more information whenever we have a plane geodesic or a straight line segment on X. Namely, the surface X must have some symmetry.

Theorem 8.2 (Reflection and Rotation Theorems) *If a plane geodesic which is not a straight line segment lies on a minimal surface, then reflection in the plane of the geodesic is a congruence of the minimal surface.*

If a straight line segment lies on a minimal surface, then 180°*-rotation around the straight line is a a congruence of the minimal surface.*

Proof. Let $X \circ r$ be a plane geodesic but not a straight line segment on X. By a rotation in $\mathbf{R}^3$ we can assume that $X \circ r$ is in the xz-plane. Since $X \circ r$ is a geodesic and is not a straight line segment, the Gauss map N of X along r must be in the xz-plane. Thus $g = \tau \circ N$ is real along r. Select a point $r(t_0)$ such that $g'(r(t_0)) \neq 0$; then in a simply connected neighbourhood U of $r(t_0)$, $w = g(z)$ is a well defined coordinate of M. We can use the representation (6.27) and consider the holomorphic mapping on U,

$$(G^1, G^2, G^3) = G := \int_{p_0}^{p} (\omega_1,\, \omega_2,\, \omega_3) + C,$$

where $p_0 = r(t_0)$ and $C \in \mathbf{C}^3$ is a constant complex vector. Remember that our surface $X = \Re G$ and $X_{\pi/2} = -\Im G$. By Proposition 8.1, $X_{\pi/2} \circ r$ is a straight line segment. Since the Gauss map of $X_{\pi/2}$ is the same as that of X, the Gauss map of $X_{\pi/2}$ is in the xz-plane along r, so the straight line segment $X_{\pi/2} \circ r$ is parallel to the y-axis. Thus $\Im G^1 \circ r$ and $\Im G^3 \circ r$ are constants. By adjusting C we may assume that the constants are zeros. Remember that along r, $w \in \mathbf{R}$. Now let $U_+ := \{w \in U \,|\, \Re w \geq 0\}$ and $U_- := \{w \in U \,|\, \Re w \leq 0\}$. We can extend $G^1\big|_{U_+}$ and $G^3\big|_{U_+}$ to U by $\tilde{G^i}(w) = \overline{G^i(\overline{w})}$, for $i = 1, 3$, $w \in U_-$ and $\overline{w} \in U_+$. Since $\Re G^2 \circ r = 0$, we can extend $G^2\big|_{U_+}$ to U_- by $\tilde{G}^2(w) = -\overline{G^2(\overline{w})}$, for $w \in U_-$ and $\overline{w} \in U_+$. Since G is holomorphic, we know that $\tilde{G}$ is holomorphic and $\tilde{G} = G$ on U. Choose a small disk $D \subset U_- \cup U_+$ such that $\overline{D} = D$, then $Y = \Re \tilde{G}$ is a minimal surface on D. Since $X = Y$ on $D \cap U_-$, by the Extension Theorem (Theorem 4.2), $X = Y$ on D. Looking at the real part, we have for any $w \in D$,

$$(X^1(w),\, X^2(w),\, X^3(w)) = \Re G(w) = \Re \tilde{G}(w) = \left(X^1(\overline{w}),\, -X^2(\overline{w}),\, X^3(\overline{w})\right) = X(\overline{w}),$$

which is a reflection in the xz-plane. By the Extension Theorem (Theorem 4.2) again, this reflection is a congruence of X.

Similarly we can prove that if $X \circ r$ is a straight line segment, then the rotation by $180°$ around $X \circ r$ is a congruence of X. □

Exercise : Prove that if $X \circ r$ is a straight line segment, then rotation by $180°$ around $X \circ r$ is a congruence of X.

Finally, we show that each component of the Gauss map N is an eigenvector of the Laplace operator $\triangle_X$. First remember that for a conformal representation of a minimal surface, $\triangle_X = \Lambda^{-2} \triangle$.

Proposition 8.3 *The Gauss map N satisfies*

$$\triangle_X N = 2KN, \tag{8.36}$$

where K is the Gauss curvature.

Proof. Let g and η be the Enneper-Weierstrass data for X. On an isothermal neighbourhood (U, z) we have

$$\begin{aligned} \triangle N &= 4\frac{\partial^2}{\partial z \partial \overline{z}} N = 4\frac{\partial^2}{\partial z \partial \overline{z}} \left[\frac{1}{1+|g|^2}\left(2\Re g, 2\Im g, |g|^2 - 1\right)\right] \\ &= \left[4\frac{\partial^2}{\partial z \partial \overline{z}}(1+|g|^2)^{-1}\right]\left(2\Re g, 2\Im g, |g|^2 - 1\right) + 4(1+|g|^2)^{-1}\frac{\partial^2}{\partial z \partial \overline{z}}\left(2\Re g, 2\Im g, |g|^2 - 1\right) \\ &\quad +8\Re\left\{\left[\frac{\partial}{\partial z}(1+|g|^2)^{-1}\right]\frac{\partial}{\partial \overline{z}}\left(2\Re g, 2\Im g, |g|^2 - 1\right)\right\}. \end{aligned}$$

Since g is holomorphic,

$$\frac{\partial}{\partial z}(1+|g|^2)^{-1} = \frac{-g'\overline{g}}{(1+|g|^2)^2},$$

$$4\frac{\partial^2}{\partial z\partial \overline{z}}(1+|g|^2)^{-1} = \frac{4|g'|^2(|g|^2-1)}{(|g|^2+1)^3}.$$

Using the Cauchy-Riemann equations we have

$$\frac{\partial}{\partial \overline{z}}\left(2\Re g, 2\Im g, |g|^2-1\right) = (\overline{g'}, i\overline{g'}, g\overline{g'}).$$

Sincce $\Re g$ and $\Im g$ are harmonic,

$$\frac{\partial^2}{\partial z\partial \overline{z}}\left(2\Re g, 2\Im g, |g|^2-1\right) = (0,0,|g'|^2).$$

Hence

$$\begin{aligned}
\triangle N &= \frac{4|g'|^2(|g|^2-1)}{(1+|g|^2)^3}\left(2\Re g, 2\Im g, |g|^2-1\right) + 4(1+|g|^2)^{-1}(0,0,|g'|^2) \\
&\quad +8\Re\left[\frac{-g'\overline{g}}{(1+|g|^2)^2}\overline{g'}\,(1,i,g)\right] \\
&= \frac{4|g'|^2(|g|^2-1)}{(1+|g|^2)^3}\left(2\Re g, 2\Im g, |g|^2-1\right) + 4(1+|g|^2)^{-1}(0,0,|g'|^2) \\
&\quad +8\left[\frac{-|g'|^2\overline{g}}{(1+|g|^2)^2}\left(\Re g, \Im g, |g|^2\right)\right] \\
&= \frac{4|g'|^2}{(|g|^2+1)^2}\frac{|g|^2}{1+|g|^2}\left(2\Re g, 2\Im g, |g|^2-1\right) + \frac{4|g'|^2}{(1+|g|^2)^2}\frac{1}{1+|g|^2}(0,0,(1+|g|^2)^2) \\
&\quad -\frac{4|g'|^2}{(1+|g|^2)^2}\frac{1+|g|^2}{1+|g|^2}\left(2\Re g, 2\Im g, 2|g|^2\right)] \\
&= \frac{-8|g'|^2}{(1+|g|^2)^2}\left[\frac{1}{1+|g|^2}\left(2\Re g, 2\Im g, |g|^2-1\right)\right] = \frac{-8|g'|^2}{(1+|g|^2)^2}N.
\end{aligned}$$

Now by (7.28) and (7.30),

$$K\Lambda^2 = \frac{-4|g'|^2}{(1+|g|^2)^2},$$

and thus

$$\triangle N = 2K\Lambda^2 N.$$

□