6 The Enneper-Weierstrass Representation

Suppose that $X : M \hookrightarrow \mathbf{R}^3$ is minimal. Since X is harmonic, on an isothermal neighbourhood $(U, (x, y))$,

$$\phi = (\phi_1, \phi_2, \phi_3) = \frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y} = 2\frac{\partial X}{\partial z} \tag{6.15}$$

is holomorphic. In fact,

$$\frac{\partial \phi}{\partial \bar{z}} = 2\frac{\partial^2 X}{\partial \bar{z}\partial z} = \frac{1}{2} \triangle X = \vec{0}.$$

Let V be another isothermal neighborhood with coordinate $w = u + iv$, and let

$$\tilde{\phi} = \frac{\partial X}{\partial u} - i\frac{\partial X}{\partial v}.$$

On $U \cap V$

$$\begin{aligned} \phi = \frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y} &= \frac{\partial X}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial x} - i\left(\frac{\partial X}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial y}\right) \\ &= \left(\frac{\partial X}{\partial u} - i\frac{\partial X}{\partial v}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) = \tilde{\phi}\frac{dw}{dz}. \end{aligned} \tag{6.16}$$

Hence

$$\tilde{\phi}\, dw = \phi\, dz, \tag{6.17}$$

which means that $\phi\, dz$ gives a globally defined vector valued holomorphic 1-form. Write

$$\omega = (\omega_1, \omega_2, \omega_3) = (\phi_1, \phi_2, \phi_3)dz = \phi\, dz. \tag{6.18}$$

By the definition of ϕ, X being conformal is equivalent to

$$\sum_{i=1}^{3} \omega_i^2 = \sum_{i=1}^{3} \phi_i^2 (dz)^2 = 0. \tag{6.19}$$

The condition that X is an immersion is equivalent to

$$\infty > \sum_{i=1}^{3} |\omega_i|^2 = \sum_{i=1}^{3} |\phi_i|^2 |dz|^2 = \left(\left|\frac{\partial X}{\partial x}\right|^2 + \left|\frac{\partial X}{\partial y}\right|^2\right) |dz|^2 = 2\Lambda^2 |dz|^2 > 0. \tag{6.20}$$

Remark 6.1 When $\sum_{i=1}^{3} |\omega_i|^2 = 0$ at some point $p \in M$, we call p a *branch point* of the surface $X : M \to \mathbf{R}^3$. At such a point, X ceases to be an immersion. At times we want to study minimal surfaces with branch points, called *branched minimal surfaces.* For branched minimal surface, since our data ϕ is holomorphic, we see that branch points are isolated. Thus in any precompact domain there are at most a finite number of branch points.

Our main interest is in minimal surfaces without branch points. All minimal surfaces in these notes are branch point free, unless specified otherwise.

The immersion X can be expressed as

$$X(p) = X(p_0) + \Re \int_{p_0}^{p} \omega, \tag{6.21}$$

where p_0 is a fixed point of M. For any closed path γ on M,

$$\Re \int_{\gamma} \omega = (0,0,0), \tag{6.22}$$

since X is well defined.

On the other hand, if we have three holomorphic 1-forms ω_i on M satisfying (6.19), (6.20), and (6.22) for any closed path γ in M, then (6.21) gives a minimal surface. This is because as the real part of a holomorphic mapping, X is harmonic; (6.19) is equivalent to X being conformal; (6.20) says that X is an immersion; and (6.22) guarantees that X is well defined.

So far everything we discussed in these notes is true in case $X : M \hookrightarrow \mathbf{R}^n$, $n \geq 3$, except the minimal surface equation should be a system of equations for $n > 3$ and the theorem about equiangular systems. Here is something special to the case $n = 3$. Let us write (6.19) as

$$(\omega_1 - i\omega_2)(\omega_1 + i\omega_2) + \omega_3^2 = 0. \tag{6.23}$$

We can assume that $\omega_3 \not\equiv 0$, as otherwise the surface lies in a plane parallel to the xy-plane, and by rotation we can get an equivalent surface such that $\omega_3 \not\equiv 0$. We define a meromorphic function g on M by

$$g = \frac{\omega_3}{\omega_1 - i\omega_2} \not\equiv 0.$$

By (6.23),

$$g^2 = \frac{\omega_3^2}{(\omega_1 - i\omega_2)^2} = -\frac{\omega_1 + i\omega_2}{\omega_1 - i\omega_2}.$$

Writing $\eta = \omega_1 - i\omega_2$, after a little calculation we have

$$\begin{cases} \omega_1 &= \frac{1}{2}(1 - g^2)\eta, \\ \omega_2 &= \frac{i}{2}(1 + g^2)\eta, \\ \omega_3 &= g\eta. \end{cases} \tag{6.24}$$

Then (6.21) can be written as

$$X(p) = X(p_0) + \Re \int_{p_0}^{p} \left(\frac{1}{2}(1 - g^2)\eta,\ \frac{i}{2}(1 + g^2)\eta,\ g\eta \right). \tag{6.25}$$

The formula (6.25) is called the *Enneper-Weierstrass representation* of the minimal surface $X : M \hookrightarrow \mathbf{R}^3$.

The meromorphic function g and the holomorphic 1-form η are called the *Enneper-Weierstrass data* of the minimal surface X, or shortly the *data* of X.

It is convenient in local coordinates to write $\eta = f(z)dz$, where $z = x + iy$ and f is a holomorphic function. Thus (6.24) can be written as

$$\begin{cases} \omega_1 &= \frac{1}{2}f(1-g^2)dz \\ \omega_2 &= \frac{i}{2}f(1+g^2)dz \\ \omega_3 &= fg\,dz. \end{cases} \tag{6.26}$$

Since g is a meromorphic function, if $dg \neq 0$ and g is not a pole at $p \in M$, then g is a holomorphic diffeomorphism in a neighbourhood U of p. Suppose U is a coordinate neighbourhood, with coordinate $z = x + iy$. Then $w = u(z) + iv(z) = g(z)$ is a local coordinate as well, and $dw = g'(z)dz = g' \circ g^{-1}(w)dz$. We define

$$F(w) = \frac{f \circ g^{-1}(w)}{g' \circ g^{-1}(w)}, \quad F(w)dw = f \circ g^{-1}(w)dz = f(z)dz = \eta.$$

Hence in the w coordinate, (6.26) becomes

$$\begin{cases} \omega_1 &= \frac{1}{2}(1-w^2)F(w)dw \\ \omega_2 &= \frac{i}{2}(1+w^2)F(w)dw \\ \omega_3 &= F(w)w\,dw. \end{cases} \tag{6.27}$$

The function F is called the *Weierstrass function* of the minimal surface $X \circ g^{-1} : g(U) \hookrightarrow \mathbf{R}^3$, where $g(U) \subset \mathbf{C}$ is a domain in $\mathbf{C}$. Notice that this is only a local representation which holds as long as g is a holomorphic diffeomorphism on U.

Now let us analyse (6.20). By (6.24), (6.20) is true if and only if whenever g has a pole of order m at $p \in M$, then η has a zero of order $2m$ at $p \in M$. Moreover, this is the only case where η can vanish.

In summary, if we have a meromorphic function g and a holomorphic 1-form η on M, such that (6.24) defines three holomorphic 1-forms which satisfy (6.20) and (6.22), then (6.25) defines a minimal surface. An important fact is that recently many interesting minimal surfaces were discovered via the Enneper-Weierstrass representation by specifying g and η on certain Riemann surfaces. See, for example, [31], [39], [41], and [80].