4 The Minimal Surface Equation

Sometimes our surface is a graph over a domain $\Omega \subset \mathbf{R}^2$, i.e., $(x, y, z) \in X(M)$ is expressed as $z = z(x, y)$, $(x, y) \in \Omega$. Moreover, locally we can always treat a "small piece" of surface as a graph. Thus we need know the differential equation governing z, the *minimal surface equation*, in order to derive more information.

To derive the minimal surface equation we use the following equivalent form of $\triangle_X$,

$$\triangle_X X = \sum_{i=1}^{2} [\tau_i \tau_i X - (\nabla_{\tau_i} \tau_i) X], \tag{4.7}$$

where $(\tau_1, \tau_2)(p)$ is an orthonormal frame of $T_p M$ in the induced metric by X and $\nabla_{\tau_i} \tau_i = (D_{\tau_i} \tau_i)^T$ is the covariant differential, in our case, namely the tangent part of $D_{\tau_i} \tau_i$.

Our surface can be written as

$$X(x, y) = (x, y, z(x, y)), \quad (x, y) \in \Omega.$$

Thus $X_x = (1, 0, z_x)$ and $X_y = (0, 1, z_y)$. We will take the upward normal

$$N = \frac{1}{(1 + z_x^2 + z_y^2)^{1/2}} (-z_x, -z_y, 1).$$

We take (τ_1, τ_2) as

$$\tau_1 = dX^{-1} \left(\frac{1}{(1 + z_x^2)^{1/2}} X_x \right) = \frac{1}{(1 + z_x^2)^{1/2}} \frac{\partial}{\partial x},$$

$$\tau_2 = dX^{-1} \left[\left(\frac{1 + z_x^2}{1 + z_x^2 + z_y^2} \right)^{1/2} \left(X_y - \frac{z_x z_y}{1 + z_x^2} X_x \right) \right] = \left(\frac{1 + z_x^2}{1 + z_x^2 + z_y^2} \right)^{1/2} \left(\frac{\partial}{\partial y} - \frac{z_x z_y}{1 + z_x^2} \frac{\partial}{\partial x} \right).$$

By (4.7) and (2.1),

$$\begin{aligned} 2H &= \left(D_{\tau_1} \frac{1}{(1 + z_x^2)^{1/2}} \frac{\partial}{\partial x} + D_{\tau_2} \left[\left(\frac{1 + z_x^2}{1 + z_x^2 + z_y^2} \right)^{1/2} \left(\frac{\partial}{\partial y} - \frac{z_x z_y}{1 + z_x^2} \frac{\partial}{\partial x} \right) \right] \bullet N \\ &= \left[\frac{X_{xx}}{1 + z_x^2} + \frac{z_x^2 z_y^2 X_{xx}}{(1 + z_x^2)(1 + z_x^2 + z_y^2)} - \frac{2 z_x z_y X_{xy}}{1 + z_x^2 + z_y^2} + \frac{(1 + z_x^2) X_{yy}}{1 + z_x^2 + z_y^2} \right] \bullet N \\ &= \left[\frac{1 + z_y^2}{1 + z_x^2 + z_y^2} X_{xx} - \frac{2 z_x z_y}{1 + z_x^2 + z_y^2} X_{xy} + \frac{1 + z_x^2}{1 + z_x^2 + z_y^2} X_{yy} \right] \bullet N. \end{aligned}$$

Since $X_{xx} = (0, 0, z_{xx})$, $X_{xy} = (0, 0, z_{xy})$, and $X_{yy} = (0, 0, z_{yy})$, we have

$$2H = \frac{1}{(1 + z_x^2 + z_y^2)^{3/2}} \left[(1 + z_y^2) z_{xx} - 2 z_x z_y z_{xy} + (1 + z_x^2) z_{yy} \right].$$

This can be written as

$$2H = \mathbf{Div}\frac{Dz}{(1+|Dz|^2)^{1/2}} = \frac{\partial}{\partial x}\frac{z_x}{(1+z_x^2+z_y^2)^{1/2}} + \frac{\partial}{\partial y}\frac{z_y}{(1+z_x^2+z_y^2)^{1/2}}.$$

We get the minimal surface equation

$$(1+z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1+z_x^2)z_{yy} = 0, \tag{4.8}$$

or

$$\mathbf{Div}\frac{Dz}{(1+|Dz|^2)^{1/2}} = 0. \tag{4.9}$$

In general, if $H = H(x,y)$ is a given function, then the *prescribed mean curvature equation* is defined as

$$(1+z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1+z_x^2)z_{yy} = 2H(1+z_x^2+z_y^2)^{3/2}, \tag{4.10}$$

or

$$\mathbf{Div}\frac{Dz}{(1+|Dz|^2)^{1/2}} = 2H. \tag{4.11}$$

Equations (4.8) and (4.10) are second order elliptic equations. We will see that they play an important role in the study of minimal, or more generally, constant mean curvature surfaces.

For example, let $\Omega \subset \mathbf{R}^2$ be a C^2 simply connected domain, $\phi \in C^0(\partial\Omega)$. Then $(x, \phi(x))$ defines a Jordan curve (continuously embedded closed curve) Γ in $\mathbf{R}^3$, where $x \in \partial\Omega$. We want to find a minimal surface bounded by Γ. So consider the Dirichlet problem

$$\begin{cases} (1+u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2)u_{yy} = 0, & \text{in } \Omega; \\ u|_{\partial\Omega} = \phi, & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

A solution of (4.12) will give us a minimal graph, which is a minimal surface bounded by Γ. From the theory of PDE we know that

Theorem 4.1 *The Dirichlet problem (4.12) is solvable for arbitrary $\phi \in C^0(\partial\Omega)$ if and only if Ω is convex.*

See for example, [21], Theorem 16.8.

A very important problem in minimal surface theory is the *Plateau problem* which asks: is there a simply connected minimal surface bounded by a given Jordan curve Γ? In general there are alway solutions to the Plateau problem as long as Γ is *rectifiable*, that is, has finite arc length. We are not going to discuss the Plateau problem in these notes.

There is a general theorem which says that for certain elliptic equations (including the minimal surface equation) the solution is real analytic. A simple proof of this fact for the minimal surface equation (2-dimensional) can be found in [61], §131 on page 125. The proof there uses special isothermal coordinates (see the next section), which shows that for the minimal surface, we do not need to call on the classical isothermal coordinate theorem.

One application of real analyticity is that if two minimal surfaces coincide in a piece of surface, then they must be essentially the same.

Theorem 4.2 (Extension Theorem) *Suppose $X : M \hookrightarrow \mathbf{R}^3$ and $Y : N \hookrightarrow \mathbf{R}^3$ are two connected minimal surfaces. If there are open sets $U \subset M$ and $V \subset N$ such that $X(U) = Y(V)$, then $X(M) \cup Y(N)$ is contained in a (perhaps larger) minimal surface.*

Proof. We prove that $X(M) \cup Y(N)$ is an immersed surface. To prove this we define $A \subset \overline{X(M)} \cap \overline{Y(N)}$ such that $x \in A$ if and only if there is a small ball B in $\mathbf{R}^3$ centred at x and either $X(M) \cap B \subset Y(N) \cap B$ or $Y(N) \cap B \subset X(M) \cap B$. By our hypothesis, $A \neq \emptyset$. We need only prove that A is closed in $\overline{X(M)} \cap \overline{Y(N)}$, since then clearly $A \cup X(M)$ and $A \cup Y(N)$ are both immersed surfaces.

First assume that X and Y are embedded.

If A is not closed, then there is a point $p \in (\overline{A} - A) \cap \overline{X(M)} \cap \overline{Y(N)}$. Thus there is a sequence $\{x_n\} \subset A$ such that $\lim_{n\to\infty} x_n = p$ and $p \in \overline{X(M)} \cap \overline{Y(N)}$. By definition of A, locally $X(M)$ and $Y(N)$ coincide at x_n, hence $X(M)$ and $Y(N)$ have the same tangent plane at x_n. Taking limits, we know that $X(M)$ and $Y(N)$ have the same limit tangent plane at $p \in \overline{X(M)} \cap \overline{Y(N)}$. After a rotation and translation if necessary, we can assume that $p = (0,0,0)$ and the common tangent plane of $X(M)$ and $Y(N)$ at p is the xy-plane. Then in a small disk D in the xy-plane centred at $(0,0)$, $X(M)$ and $Y(N)$ are graphs over domains $\Omega_1 \subset D$ and $\Omega_2 \subset D$ such that $(0,0) \in \overline{\Omega_1} \cap \overline{\Omega_2}$. Thus there are u and v satisfying the minimal surface equation on Ω_1 and Ω_2 respectively, such that $(x, y, u(x, y))$ represents $X(M)$ and $(x, y, v(x, y))$ represents $Y(N)$. By definition of p, we know that there is an open subset $Q \subset$ of $\Omega_1 \cap \Omega_2$ on which $u \equiv v$. But u and v are real analytic, so $u \equiv v$ on $\Omega_1 \cap \Omega_2$. Hence both u and v can be extended to $\Omega_1 \cup \Omega_2$, and represent the same surface. This is a contradiction to the assumption $p \notin A$. Thus A is closed in $\overline{X(M)} \cap \overline{Y(N)}$.

If X or Y is not an embedding, first consider the local version of the proof, then modify the definition of A at multiple points of $\mathbf{R}^3$, i.e., at points which are images of more than one point of M or of N.

The proof then is complete. □

Definition 4.3 An *equiangular system of order k* at a point $q \in \mathbf{C}$ consists of k curved rays $\gamma_1, \gamma_2, \cdots, \gamma_k$ emitting from q such that any two adjacent rays intersect at q with angle $2\pi/k$.

Theorem 4.4 *Let $X : M \hookrightarrow \mathbf{R}^3$ and $Y : N \hookrightarrow \mathbf{R}^3$ be two minimal surfaces and $x \in X(M) \cap Y(N)$ be such that $X(M)$ and $Y(N)$ at x have the same tangent plane P. Then* *either $X(M)$ and $Y(N)$ are part of a (maybe larger) minimal surface or the orthogonal projection of $X(M) \cap Y(N)$ on P forms an equiangular system of even order $k \geq 4$.*

Proof. By a rotation and translation, we may assume that $x = (0,0,0)$ and P is the xy-plane. Then there is a disk $D \subset P$ centred at $(0,0)$ such that $X(M)$ and $Y(N)$ are graphs given by $u : D \to \mathbf{R}$ and $v : D \to \mathbf{R}$ respectively. Moreover, since P is the common tangent plane, $Du = Dv = (0,0)$ at $(0,0)$.

Let $w = v - u$, then by real analyticity, w satisfies

$$w = \sum_{n=k}^{\infty} P^{(n)}(x,y), \quad k \geq 2,$$

where

$$P^{(n)}(x,y) = \sum_{i=0}^{n} \frac{1}{i!(n-i)!} \frac{\partial^n w}{\partial^i x \partial^{n-i} y}(0,0) x^i y^{n-i}$$

is a homogeneous polynomial of degree n. If $P^{(n)}(x,y) \equiv 0$ for $n \geq 2$, then $u \equiv v$ in D. By Theorem 4.2, $X(M)$ and $Y(N)$ are part of a (maybe larger) minimal surface.

If $P^{(n)}(x,y) \not\equiv 0$ for some $n \geq 2$, then let k be the smallest n such that $P^{(n)} \not\equiv 0$. In this case, we say that $X(M)$ and $Y(N)$ has $k-1$ *contact.*

Now since u and v satisfy the minimal surface equation, we have

$$\begin{aligned}
\triangle w &= \triangle v - \triangle u \\
&= 2v_x v_y v_{xy} - 2u_x u_y u_{xy} - v_y^2 v_{xx} + u_y^2 u_{xx} - v_x^2 v_{yy} + u_x^2 u_{yy} \\
&= -u_y^2 w_{xx} + (u_y^2 - v_y^2) v_{xx} - u_x^2 w_{yy} + (u_x^2 - v_x^2) v_{yy} + 2u_x u_y w_{xy} - 2(u_x u_y - v_x v_y) v_{xy} \\
&= -u_y^2 w_{xx} + (u_y + v_y)(u_y - v_y) v_{xx} - u_x^2 w_{yy} + (u_x + v_x)(u_x - v_x) v_{yy} \\
&\quad + 2u_x u_y w_{xy} - 2[v_x(u_y - v_y) + (u_x - v_x) u_y] v_{xy} \\
&= -u_y^2 w_{xx} - (u_y + v_y) v_{xx} w_y - u_x^2 w_{yy} - (u_x + v_x) v_{yy} w_x \\
&\quad + 2u_x u_y w_{xy} + 2v_x v_{xy} w_y + 2u_y v_{xy} w_x = O(r^k),
\end{aligned}$$

where $r = (x^2 + y^2)^{1/2}$. The last equality comes from the fact that $Du = Dv = (0,0)$ at $(0,0)$ and $w = O(r^k)$. By

$$\triangle P^{(n)} = O(r^{n-2}) \quad \text{and} \quad \triangle w = O(r^k),$$

we have that

$$\triangle P^{(k)} = O(r^{k-1}).$$

Since $\triangle P^{(k)}$ is a polynomial of degree at most $k-2$, it must be the case that $\triangle P^{(k)} = 0$, that is, $P^{(k)}$ is a harmonic polynomial.

Now $P^{(k)}(x,y) = \Re H(z)$, where H is a holomorphic function, $\Re$ denotes the real part, and $z = x + iy$. Since $P^{(k)} = O(r^k)$, we can choose H such that $H(z) = z^k F(z)$, where $F(0) \neq 0$. In a smaller disk contained in D, $(F(z))^{1/k}$ is well defined, hence let

$\zeta = z(F(z))^{1/k}$, then $H(z) = \zeta^k$. Let $\zeta = \rho e^{i\psi} = \xi + i\eta$, we have $P^{(k)} = \Re H(z) = \rho^k \cos(k\psi)$. Thus the zero set of $P^{(k)}$ is an equiangular system of even order $2k \geq 4$.

Since $w = P^{(k)}(x,y) + \sum_{n=k+1}^{\infty} P^{(n)}(x,y)$ is analytic and $\sum_{n=k+1}^{\infty} P^{(n)}(x,y) = o(r^k)$, the zero set of w also consists of an equiangular system.

The projection of $X(M) \cap Y(N)$ around $(0,0,0)$ on P is exactly the zero set of w. The proof of the theorem is complete. □

Corollary 4.5 *Let $X(M)$ be a non-planar minimal surface, $p \in X(M)$ and $P = T_pM \subset T_{X(p)}\mathbf{R}^3$. Then $X(M) \cap P$ consists of an equiangular system of even order at least* 4.

Proof. This is the special case that P is the minimal surface $Y(N)$. □

Remark 4.6 Theorem 4.4 and Corollary 4.5 are called *maximum* (or *comparison*) *principle for minimal surface.* Together with Theorem 4.2 it follows that two minimal surfaces cannot touch each other at isolated interior points.