Appendix A

On the "hyperboloidal initial data", and Penrose conditions.

Let us briefly recall the conformal framework introduced by Penrose [104] to describe the behaviour of physical fields at null infinity. Given a, say vacuum, smooth "physical" space-time $(\tilde{M}, \tilde{\gamma})$ one associates to it a smooth "unphysical space-time" (M, γ) and a smooth function Ω on M, such that $\tilde{M}$ is a subset of M and

$$\Omega|_{\tilde{M}} > 0 , \quad \gamma_{\mu\nu}|_{\tilde{M}} = \Omega^2 \tilde{\gamma}_{\mu\nu} , \tag{A.0.1}$$

$$\Omega|_{\partial\tilde{M}} = 0 , \tag{A.0.2}$$

$$d\Omega(p) \neq 0 \quad \text{for} \quad p \in \partial\tilde{M} , \tag{A.0.3}$$

where $\partial\tilde{M}$ is the boundary of $\tilde{M}$ in M (it should be stressed that in this section a notation *inverse* to that used in 1.6 is used: tilded quantities denote the physical ones, while non–tilded quantities denote the unphysical (conformally rescaled) ones). It is common usage in general relativity to use the symbol $\mathcal{I}$ for $\partial\tilde{M}$, and we shall sometimes do so. If Σ is a hypersurface in M, by $\mathcal{I}^+$ we shall denote the connected component of $\mathcal{I}$ which intersects the causal future of Σ. The hypothesis of smoothness of (M, γ, Ω) and the fact that $(\tilde{M}, \tilde{\gamma})$ is vacuum imposes several restrictions on various fields; if one defines (*cf.* [104])

$$P_{\mu\nu} = \tfrac{1}{2} \left(R_{\mu\nu} - \tfrac{1}{6} R \gamma_{\mu\nu} \right) , \tag{A.0.4}$$

with an analogous definition for the tilded quantities, one has

$$0 = \tilde{P}_{\mu\nu} = P_{\mu\nu} - \tfrac{1}{\Omega}\nabla_\mu \nabla_\nu \Omega + \tfrac{1}{2\Omega^2}\nabla^\alpha \Omega \nabla_\alpha \Omega \gamma_{\mu\nu} , \tag{A.0.5}$$

where ∇_μ is the covariant derivative of the metric $\gamma_{\mu\nu}$. (A.0.2) and (A.0.5) imply

$$\nabla^\alpha \Omega \nabla_\alpha \Omega|_{\partial\tilde{M}} = 0 \tag{A.0.6}$$

$$(\nabla_\mu \nabla_\nu \Omega - \tfrac{1}{4}\nabla^\alpha \nabla_\alpha \Omega \gamma_{\mu\nu})|_{\partial\tilde{M}} = 0 . \tag{A.0.7}$$

Suppose that $\Sigma \subset M$ is a spacelike hypersurface in (M,γ), let $\tilde{\Sigma} = \Sigma \cap \tilde{M}$, $\partial\Sigma = \partial\tilde{\Sigma} = \Sigma \cap \partial\tilde{M}$, and let g_{ij}, K_{ij}, respectively $\tilde{g}_{ij}, \tilde{K}_{ij}$, be the induced metric and extrinsic curvature of Σ in (M,γ), respectively $\tilde{\Sigma}$ in $(\tilde{M},\tilde{\gamma})$. If we denote by L^{ij} and $\tilde{L}^{ij}$ the traceless part of $K^{ij} = g^{ik}g^{j\ell}K_{k\ell}$, $\tilde{K}^{ij} = \tilde{g}^{ik}\tilde{g}^{j\ell}\tilde{K}_{k\ell}$,

$$\begin{aligned} L^{ij} &= K^{ij} - \tfrac{1}{3} K g^{ij} , \quad K = g^{ij} K_{ij} , \\ \tilde{L}^{ij} &= \tilde{K}^{ij} - \tfrac{1}{3} \tilde{K} \tilde{g}^{ij} , \quad \tilde{K} = \tilde{g}^{ij} \tilde{K}_{ij} , \end{aligned} \tag{A.0.8}$$

one finds

$$\begin{aligned} \tilde{L}^{ij} &= \Omega^3 L^{ij} , \quad |\tilde{L}|_{\tilde{g}} = \Omega |L|_g , \\ \tilde{K} &= \Omega K - 3 n^\alpha \Omega_{,\alpha} , \end{aligned} \tag{A.0.9}$$

where n^α is the unit normal to Σ for the metric γ, and $|\cdot|_h$ denotes the tensor norm in a Riemannian metric h. Since n^α is timelike and $\nabla\Omega(p)$ is null for $p \in \partial\tilde{\Sigma}$ we have

$$\tilde{K}|_{\partial\tilde{\Sigma}} = -3n^\alpha \Omega_{,\alpha}|_{\partial\tilde{\Sigma}} \gtrless 0 , \tag{A.0.10}$$

because the scalar product of two non–vanishing non–spacelike vectors cannot change sign. From (A.0.2) we also have

$$g_{ij}|_{\tilde{\Sigma}} = \Omega^2 \tilde{g}_{ij} ,$$

and since $\nabla\Omega$ is null non–vanishing at $\partial\tilde{\Sigma}$ the equations (A.0.9)–(A.0.10) imply

$$D^i \Omega D_i \Omega|_{\partial\tilde{\Sigma}} = \left(\frac{\tilde{K}}{3}\right)^2\Big|_{\partial\tilde{\Sigma}} > 0 , \tag{A.0.11}$$

where D_i is the Riemannian connection of the metric g_{ij}. To summarize, necessary conditions for an initial data set $(\tilde{\Sigma},\tilde{g},\tilde{K})$ to arise from an "extended initial data set (Σ, g, K) intersecting a smooth $\mathcal{I}$" are

C1. There exists a Riemannian manifold (Σ, g), $g \in C^k(\Sigma)$, such that $\tilde{\Sigma}$ is a submanifold of Σ with smooth boundary $\partial\tilde{\Sigma}$. Moreover there exists a function $\Omega \in C^k(\Sigma)$ such that

$$g_{ij}|_{\tilde{\Sigma}} = \Omega^2 \tilde{g}_{ij} , \tag{A.0.12}$$

$$\Omega|_{\partial\tilde{\Sigma}} = 0 , \qquad |D\Omega|_{g|\partial\tilde{\Sigma}} > 0 . \tag{A.0.13}$$

C2. The symmetric tensor field $\tilde{K}^{ij}$ satisfies, for some $\tilde{K} \in C^{k-1}(\Sigma)$, $\tilde{L}^{ij} \in C^{k-1}(\Sigma)$,

$$\tilde{K}^{ij} = \tilde{L}^{ij} + \tfrac{1}{3}\tilde{K}\,\tilde{g}^{ij} , \quad \tilde{K} = \tilde{g}_{ij}\,\tilde{K}^{ij} , \tag{A.0.14}$$

$$\tilde{K}|_{\partial\tilde{\Sigma}} \text{ is nowhere vanishing} , \tag{A.0.15}$$

$$|\tilde{L}|_{\tilde{g}|\partial\tilde{\Sigma}} = 0 . \tag{A.0.16}$$

If there existed "a lot" of space-times satisfying the Penrose conformal conditions, there should exist "a lot" of initial data satisfying C1–C2. It is therefore natural to ask the question, can one construct such data sets? This involves constructing solutions of the scalar constraint equation,

$$\tilde{R}(\tilde{g}) + \tilde{K}^2 - \tilde{K}_{ij}\,\tilde{K}^{ij} = 0 , \tag{A.0.17}$$

where $\tilde{R}(\tilde{g})$ is the Ricci scalar of the metric $\tilde{g}$, and the vector constraint equation,

$$\tilde{D}_i(\tilde{K}^{ij} - \tilde{K}\,\tilde{g}^{ij}) = 0 , \tag{A.0.18}$$

where $\tilde{D}$ is the Riemannian connection of the metric $\tilde{g}$, under appropriate asymptotic conditions. No general method of producing solutions of (A.0.17)–(A.0.18) is known, unless one assumes

C3.

$$\tilde{D}_i\tilde{K} \equiv 0 . \tag{A.0.19}$$

Under (A.0.19) the scalar and the vector constraint equations decouple, and the well known Choquet-Bruhat-Lichnerowicz-York conformal procedure allows one to construct solutions of (A.0.17)–(A.0.18). An initial data set satisfying C1–C3 will be called a C^k *hyperboloidal initial data set* (smooth if $k = \infty$), while conditions C1– C2 will be called Penrose's C^k conditions. Without loss of generality we may normalize $\tilde{K}$ so that

$$\tilde{K} = 3\,, \tag{A.0.20}$$

and (A.0.17)–(A.0.18) can be rewritten as

$$\tilde{R}(\tilde{g}) + 6 = \tilde{L}_{ij}\,\tilde{L}^{ij} \tag{A.0.21}$$

$$\tilde{D}_i\,\tilde{L}^{ij} = 0\,. \tag{A.0.22}$$

To construct solutions of (A.0.21)–(A.0.22) one can proceed as follows: fix a compact Riemannian manifold $(\Sigma, \mathring{g}\,)$, let $\tilde{\Sigma}$ be an open submanifold of Σ with compact closure and with smooth boundary $\partial\tilde{\Sigma}$, and let $\mathring{\Omega}$ be any defining function for $\partial\tilde{\Sigma}$ (by definition,

$$\mathring{\Omega}|_{\partial\tilde{\Sigma}} = 0\,, \qquad |d\,\mathring{\Omega}|_{\mathring{g}}\,{}_{|_{\partial\tilde{\Sigma}}} > 0\,,$$

and $\mathring{\Omega}(p) = 0 \Rightarrow p \in \partial\tilde{\Sigma}$), set

$$\bar{g}_{ij} = \mathring{\Omega}^{-2}\,\mathring{g}_{\,ij}\,.$$

Given a smooth traceless symmetric tensor field $\mathring{L}^{ij}$ on Σ satisfying

$$\bar{D}_i(\mathring{\Omega}^3\,\mathring{L}^{ij}) = 0 \Longrightarrow \mathring{D}_{\,i}(\Omega_0^{-2}\,\mathring{L}^{ij}) = 0\,, \tag{A.0.23}$$

where $\bar{D}$, $\mathring{D}$ are the Riemannian connections of the metrics $\bar{g}$, $\mathring{g}$, it is not too difficult to check that the fields

$$\tilde{g}_{ij} = \phi^4\,\bar{g}_{ij}\,,$$

$$\tilde{L}^{ij} = \phi^{-10}\bar{L}^{ij} \equiv \phi^{-10}\,\mathring{\Omega}^3\,\mathring{L}^{ij}\,,$$

will satisfy (A.0.21)–(A.0.22) if

$$8\bar{\Delta}\phi - \bar{R}\phi + \lambda\phi^{-7} - 6\phi^5 = 0\,, \tag{A.0.24}$$

$$\lambda \equiv \bar{g}_{ij}\,\bar{g}_{k\ell}\,\bar{L}^{ik}\,\bar{L}^{j\ell}\,,$$

where $\bar{\Delta} = \bar{D}^i \bar{D}_i$ is the Laplacian of the metric $\bar{g}_{ij}$. If moreover

$$\begin{aligned} \phi|_{\partial\tilde{\Sigma}} &= 1 & \text{(A.0.25)} \\ |\bar{L}|_{\bar{g}|\partial\tilde{\Sigma}} &= 0\,. & \text{(A.0.26)} \end{aligned}$$

then $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ will satisfy C1–C3. In [2] [3] the following is proved:

Theorem A.0.2 *For any smooth $(\tilde{\Sigma}, \mathring{g}, \mathring{\Omega}, \mathring{L})$ as above there exists a solution of (A.0.24)–(A.0.25), moreover:*

1. *For given $(\Sigma, \tilde{\Sigma})$ and for an open dense set (in $C^\infty(\Sigma)$ topology) of $(\mathring{g}, \mathring{\Omega}, \mathring{L})$ the function ϕ^{-2} can be extended to a C^2 function from $\tilde{\Sigma}$ to Σ, but* not *to a C^3 function on Σ (the third derivatives of any extension of ϕ will logarithmically blow up as one approaches $\partial\tilde{\Sigma}$); in particular for generic (in the above sense) triples $(\mathring{g}, \mathring{\Omega}, \mathring{L})$ the initial data set $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ will display asymptotic behaviour* incompatible *with Penrose's C^4 conditions.*

2. *There exists a "large set" of* non-generic *$(\mathring{g}, \mathring{\Omega}, \mathring{L})$ for which $\Omega \equiv \phi^{-2}\,\mathring{\Omega}$ can be smoothly extended from $\tilde{\Sigma}$ to Σ.*

It should be emphasized that in Theorem A.0.2 no hypotheses on the topology of Σ, $\tilde{\Sigma}$ and $\partial\tilde{\Sigma}$ are made, thus the resulting space-time may have a conformal boundary consisting of several connected components of varying topology (recall that *e.g.* some Robinson-Trautman space-times admit a smooth $\mathcal{I}$ the "spatial" topology of which is not a sphere). Let us also note that even considering only those data sets for which $\mathring{L} = 0$, or for which $\mathring{L}^{ij}$ vanishes on $\partial\tilde{\Sigma}$ to some desired order, point (1) above will still hold in the sense that for generic $(\mathring{g}, \mathring{\Omega})$ and $\mathring{L}$'s vanishing to some prescribed order (or even *e.g.* identically vanishing) no C^3 extensions of ϕ from $\tilde{\Sigma}$ to Σ will exist.

To complete the construction of initial data sets one also has to produce solutions of (A.0.23), the standard approach proceeds as follows: Let A^{ij} be a traceless tensor field, let X^i solve the equation

$$\mathring{D}_j(\mathring{D}^i X^j + \mathring{D}^j X^i - \frac{2}{3}\,\mathring{D}^k X_k\,\mathring{g}^{ij}) = -\mathring{D}_j A^{ij}\,. \tag{A.0.27}$$

The tensor field defined by

$$\mathring{L}^{ij} = \mathring{\Omega}^2 (A^{ij} + \mathring{D}^{\,i} X^k + \mathring{D}^{\,j} X^i - \frac{2}{3}\, \mathring{D}^{\,k} X_k\, \mathring{g}^{\,ij}) \tag{A.0.28}$$

will satisfy (A.0.23). If we want to obtain $\mathring{L}^{ij}$'s which do not necessarily vanish to second order at $\partial\tilde{\Sigma}$, we have to admit A^{ij}'s of the form

$$A^{ij} = \mathring{\Omega}^{-2}\, \mathring{A}^{\,ij}\,, \tag{A.0.29}$$

for some smooth $\mathring{A}^{\,ij}$, which gives $\mathring{D}_{\,j} A^{ij} = \mathring{\Omega}^{-3} Z^i$, for some smooth vector field Z^i, nonidentically vanishing at $\partial\tilde{\Sigma}$ for generic non-vanishing $\mathring{A}^{\,ij}$ at $\partial\tilde{\Sigma}$; note that X^i should vanish to third order at $\partial\tilde{\Sigma}$ if $\mathring{A}^{\,ij}$ vanishes to second order there). In [2] the following is established:

Theorem A.0.3 *Let $(\Sigma, \mathring{g}\,)$ be a smooth Riemannian manifold, let $\tilde{\Sigma}$ be a smooth submanifold of Σ with smooth boundary $\partial\tilde{\Sigma}$, with a defining function $\mathring{\Omega}$. Consider the problem*

$$\begin{aligned}(\Delta_{\mathring{L},\mathring{g}}\, X)^j &\equiv \mathring{D}_{\,i}(\mathring{D}^{\,i} X^j + \mathring{D}^{\,j} X^i - \frac{2}{3}\, \mathring{D}^{\,k} X_k\, \mathring{g}^{\,ij})|_{\tilde{\Sigma}} \\ &= \mathring{\Omega}^{\alpha} Y^j|_{\tilde{\Sigma}}\,,\end{aligned} \tag{A.0.30}$$

where Y^j is a smooth vector field on Σ and α — a negative integer. There exists a solution of (A.0.30) of the form

$$X = \mathring{\Omega}^{\alpha+2} X_1 + \log \mathring{\Omega}\, X_{\log} + X_0\,, \tag{A.0.31}$$

where X_1, $X_{\log}$, X_0 are smooth vector fields on Σ. For any $\alpha \neq 0$ there exists an open dense set of Y's (in a $C^\infty(\Sigma)$ topology) for which $X_{\log}|_{\partial\tilde{\Sigma}} \neq 0$. If $\alpha = -1$ then $X_{\log} = \mathring{\Omega}\, \hat{X}_{\log}$, for some smooth vector field $\hat{X}_{\log}$ on Σ. If $\alpha = 0$, $X_{\log} \equiv 0$. X is unique in the class of solutions of the form (A.0.31), with smooth X_0, X_1 and $X_{\log}$.

It should be pointed out that for generic $\mathring{A}^{\,ij}$, the source term in (A.0.27) with A^{ij} given by (A.0.29) will be generic and thus the corresponding solution X will have log terms, consequently $\mathring{L}^{ij}$ given by (A.0.28) wil be C^1 but *not* C^2 extendible from $\tilde{\Sigma}$ to Σ. A similar thing will happen when considering those $\mathring{A}^{ij}$ which vanish to order one at $\partial\tilde{\Sigma}$: generic such solutions will be C^2 but not C^3 extendible. On the other hand if $\mathring{A}^{ij}$ vanishes to order two or higher at the boundary, then the source term in (A.0.27) will be smooth, and so will be the solution X: in this case no log terms occur.

In order to obtain Cauchy data which can be used in Friedrich's stability theorem, Theorem 1.6.1, some more restrictions on $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ are needed, namely the vanishing at $\partial\tilde{\Sigma}$ of both the tensor $e_{\alpha\beta}$ defined in Theorem 1.6.1 and of the Weyl tensor. In the case of L_{ij} vanishing on $\partial\tilde{\Sigma}$, it turns out that [3] smooth extendability of the function Ω across $\partial\tilde{\Sigma}$ is a necessary condition for $e_{\alpha\beta}|_{\partial\tilde{\Sigma}} = C^{\alpha}_{\beta\gamma\delta}|_{\partial\tilde{\Sigma}} = 0$. (More precisely, under the conditions $\tilde{K}_{ij}\tilde{g}^{ij} = const$, $L_{ij}|_{\partial\tilde{\Sigma}} = 0$, one shows [3] that the condition $e_{ij}|_{\partial\tilde{\Sigma}} = 0$ is equivalent to the fact that the extrinsic curvature of $\partial\tilde{\Sigma}$ in Σ is pure trace; this then implies that Ω is smooth up to boundary, and that $C^{\alpha}_{\beta\gamma\delta}|_{\partial\tilde{\Sigma}} = 0$, moreover Ω_n and Ω_{nn} can be chosen so that $e_{\alpha\beta}|_{\partial\tilde{\Sigma}} = 0$ holds.) Point 1 of Theorem A.0.2 thus shows, that generic data constructed by the conformal method will not be regular enough to be used in Friedrich's existence theorems. In fact the problem here is much more serious than just being one or two degrees of differentiability away from a threshold, because one of the fields used in Friedrich's "conformally regular system" is $d^{\alpha}{}_{\beta\gamma\delta} \equiv \Omega^{-1}C^{\alpha}{}_{\beta\gamma\delta}$, where $C^{\alpha}{}_{\beta\gamma\delta}$ is the Weyl curvature tensor of the space–time metric, evaluated formally from the Cauchy data (g, K) assuming vacuum Einstein equations. Whenever $C^{\alpha}{}_{\beta\gamma\delta}(p) \not\equiv 0$ for $p \in \partial\tilde{\Sigma}$, the field $d^{\alpha}{}_{\beta\gamma\delta}$ blows up at $\partial\tilde{\Sigma}$ as $1/\Omega$, and is thus not even in $L^1(\Sigma)$. It should be stressed that nevertheless point 2 of Theorem A.0.2 establishes existence of a large class of non–trivial data with asymptotic behaviour compatible with the Penrose–Friedrich conditions.