MEAN CURVATURE EVOLUTION OF SPACELIKE HYPERSURFACES

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1. Spacelike hypersurfaces of prescribed mean curvature

Spacelike hypersurfaces of prescribed mean curvature have played an important role in the study of the structure of Lorentzian manifolds. Examples include the singularity theorems of Hawking and Penrose [HE], the analysis of the Cauchy problem for Einstein's equations based on $3+1$ foliations ([CBY], [LA]) and the first proof of the positive mass theorem due to Schoen and Yau ([SY]).

General existence and regularity results for prescribed mean curvature hypersurfaces were obtained by Gerhardt ([CG]) and Bartnik ([B1]), boundary values problems were treated in [BS] and in [B2]. For an excellent survey of the area we refer to the article by Bartnik ([B3]).

In [EH1] and [E1,2], spacelike hypersurfaces of prescribed mean curvature were constructed as stationary limits of mean curvature flow, which had previously been very successfully employed in Riemannian manifolds (see [H1] for a survey). In this talk, we will review some of the central features of this nonlinear evolution process in the special case of spacelike hypersurfaces in Minkowski space as most of the essential analytical difficulties already arise in this simplest situation.

Minkowski space $\mathbf{R}^{n,1}$ is $\mathbf{R}^{n+1}$ endowed with the metric $\langle \cdot, \cdot \rangle$ defined by $\langle X, Y \rangle = x \cdot y - x_0 y_0$ for vectors $X = (x, x_0), Y = (y, y_0)$. With regard to this metric, vectors in $\mathbf{R}^{n,1}$ can be divided into spacelike, timelike and null-vectors depending on whether they satisfy $\langle X, X \rangle > 0$, $\langle X, X \rangle < 0$ or $\langle X, X \rangle = 0$. The timelike vectors can be divided further in the natural way into future and past directed vectors.

A hypersurface $M \subset \mathbf{R}^{n,1}$ is called spacelike if it admits an everywhere timelike normal field which we assume to be future directed and to satisfy the condition $\langle \nu, \nu \rangle = -1$. Note that the metric on M induced from $\mathbf{R}^{n,1}$ is Riemannian. Spacelike hypersurfaces can locally be expressed as graphs of functions $u : \Omega \to \mathbf{R}$ satisfying $|Du(x)| < 1$ for all $x \in \Omega$ where Ω is an open subset of $\mathbf{R}^n$. In particular, a spacelike hypersurface satisfies the inequality $|u(x) - u(y)| < |x - y|$ for all $x, y \in \Omega$. In this talk we will concentrate on so-called entire graphs i.e. we will assume $\Omega = \mathrm{R}^n$ unless otherwise stated.

We define the Lorentz distance function by $z = \langle X, X \rangle$. When restricted to M this is given by $|x|^2 - u^2(x)$ which is nonnegative if M contains the origin. One can check using the mean value theorem that z is a proper function on M in the sense that $\lim_{|x|\to\infty}(|x|^2 - u^2(x)) = \infty$. This implies in particular that the sets $M \cap \{z \le \rho^2\}$ are compact in M for every $\rho > 0$.

The future directed unit normal can be expressed in terms of u by

$$\nu(x) = \frac{(Du(x), 1)}{\sqrt{1 - |Du(x)|^2}}.$$

The mean curvature of M is defined as $H = \operatorname{div} \nu$. The equation of a hypersurface with mean curvature prescribed as a function $\mathcal{H} = \mathcal{H}(\cdot, u, Du)$ can therefore be written as

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right)(x) = \mathcal{H}(x, u(x), Du(x)). \tag{1}$$

This is an elliptic partial differential equation with ellipticity constant controlled by the gradient function

$$v \equiv -\langle \nu, e_0 \rangle = \frac{1}{\sqrt{1 - |Du|^2}}.$$

One generally aims to bound v as this implies uniform ellipticity of equation (1) and therefore reduces the analysis to standard techniques for such equations.

Maximal hypersurfaces. Spacelike hypersurfaces with mean curvature $H = 0$ are called maximal hypersurfaces in view of the fact that the equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) = 0$$

is the Euler-Lagrange equation for the area functional of a spacelike hypersurface

$$\int_\Omega \sqrt{1 - |Du|^2}$$

which in $\mathbf{R}^{n,1}$ one would like to maximise subject to boundary conditions.

Theorem ([C], [CY]). *All entire solutions of the maximal surface equation for spacelike hypersurfaces in Minkowski space are spacelike hyperplanes that is graphs of functions of the form*

$$u(x) = a \cdot x + b$$

where $a \in \mathbf{R}^n$ satisfies $|a| < 1$ and $b \in \mathbf{R}$.

Constant mean curvature hypersurfaces. Examples of hypersurfaces with constant mean curvature equal to $\frac{n}{\rho}$ are the hyperboloids given by

$$u_\rho(x) = \sqrt{|x|^2 + \rho^2}.$$

In particular, the identity $z = -\rho^2$ holds on these. Note that for $\rho = 0$ we obtain the upper light cone at 0 and for $|x| \to \infty$ the hyperboloids are all asymptotic to this light cone. Other examples of constant mean curvature hypersurfaces were constructed by Treibergs [T].

An interesting property of entire spacelike hypersurfaces of constant mean curvature is that they are automatically geodesically complete. This is a consequence of the following a priori gradient estimate due to Cheng and Yau ([CY]) which is the most important ingredient in the proof of the above theorem for maximal hypersurfaces.

Theorem ([CY]). *Let $M = \operatorname{graph} u$ be an entire spacelike hypersurface of constant mean curvature H. Suppose that $0 \in M$. Then the Lorentz distance function on M satisfies the estimate*

$$(*) \qquad |\nabla \sqrt{z}| \leq C(n, H)$$

where ∇ denotes the tangential gradient to M.

Integrating this gradient estimate along geodesic rays in M and using the compactness of the sets $M \cap \{z \leq \rho^2\}$ one can show that geodesic balls in M are compact which establishes the geodesic completeness of M.

Estimate (*) can also be interpreted as a local estimate for the gradient function v but such estimates can be obtained more directly (see [B3]). We will return to this later when we discuss parabolic equations for spacelike hypersurfaces.

Geodesically incomplete spacelike hypersurfaces. Let us consider spacelike hypersurfaces which satisfy $H = v$. These arise naturally in a heat flow problem related to equation (1). The differential equation (1) in the case $\mathcal{H} = v$ can be written as

$$(2) \qquad Hv^{-1} = \sqrt{1 - |Du|^2} \operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 1.$$

For n=1 this becomes

$$\frac{u''}{1-(u')^2} = (\operatorname{arctanh} u')' = 1$$

which has $u(x) = \log\cosh x$ as a particular solution. This solution is asymptotic to the upper light cone $|x| - \log 2$ for $|x| \to \infty$ just as the constant mean curvature hyperboloids. However, unlike these its graph is geodesically incomplete, i.e. the curve has finite length in Minkowski space. In fact, one could ask whether entire solutions of (2) in any dimension are geodesically incomplete. Noting that the function $u(x) = \log\cosh|x|$ and suitable rescales of it furnish sub- and supersolutions of (2), the following existence result in higher dimensions is proved in [E2]:

Proposition. *There exist entire spacelike solutions $u : \mathbf{R}^n \to \mathbf{R}$ of equation (2).*

One can construct examples of solutions of (2) in some other simple spacetimes. In a general spacetime, the gradient function is given by $v = -\langle T, \nu \rangle$ where T is the future directed unit timelike normal to a fixed reference slicing of the spacetime. Consider for instance the Schwarzschild spacetime in 1+1 dimension with metric given by $ds^2 = -\alpha^2 dx_0^2 + \alpha^{-2} dr^2$ with $\alpha^2 = 1 - \frac{2m}{r}$ where x_0 denotes the time coordinate, $m > 0$ is the mass and $r > 2m$. One calculates (see e.g. [B1]) that

$$Hv^{-1} = \alpha \left(\operatorname{arctanh} \alpha^2 u'\right)' + \alpha^2 \alpha' u'$$

where ' denotes derivatives with respect to r. The radial null geodesics satisfy the equation $r_* = r + 2m \log \frac{r}{2m}$ (see [W]). It is interesting to note that the function $u(r) = \log \cosh r_*$ satisfies equation (2) up to lower order terms although both H and v grow exponentially for $r \to \infty$. In fact, one easily checks that

$$Hv^{-1} = \alpha^{-1} \left(1 + \frac{m}{r^2} \tanh r_*\right) = (1 + O(r^{-1})).$$

We furthermore observe that $H \geq v$. By suitably rescaling u we can find a function for which $H < v$. Similar functions can be constructed in higher dimensions and are again useful as barriers in the construction of solutions of (2).

2. A heat flow for spacelike hypersurfaces

We consider a family of spacelike embeddings $X_t = X(\cdot, t) : \mathbf{R}^n \to \mathbf{R}^{n,1}$ with corresponding hypersurfaces $M_t = X_t(\mathbf{R}^n)$ satisfying the evolution equation

$$\frac{\partial X}{\partial t} = (H - \mathcal{H})\nu \tag{3}$$

on some time interval. Here, $H = \operatorname{div}_{M_t} \nu$ denotes the mean curvature of the hypersurface M_t and $\mathcal{H}$ depends on the position and the normal vector of M_t. In view of the geometric identity $\Delta X = H\,\nu$ where Δ denotes the Laplace-Beltrami operator on M_t, equation (3) can also be written as

$$\frac{\partial X}{\partial t} = \Delta X - \mathcal{H}\,\nu \tag{3'}$$

which exhibits the parabolic structure of equation (3). Each M_t is the graph of a function $u(\cdot, t)$ satisfying $|Du(\cdot, t)| < 1$. In fact, u is equivalent up to a tangential diffeomorphism of $\mathbf{R}^n$ to the function $-\langle X, e_0 \rangle$ and equation (3) is therefore equivalent to the equation

$$\frac{\partial u}{\partial t} = \sqrt{1 - |Du|^2} \left(\operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) - \mathcal{H}(\cdot, u, Du) \right) \tag{4}$$

This is the parabolic analogue of the prescribed mean curvature equation (1).

Note in particular that spacelike hypersurfaces of prescribed mean curvature $\mathcal{H}$ arise as stationary limits of equation (4). This fact was used in [EH1] and [E1] to give new existence proofs for such hypersurfaces in a very large class of spacetimes based entirely on heat flow techniques. The existence proofs in [G] and [B1] instead rely on topological fixed point arguments.

For $\mathcal{H} = 0$, equation (4) arises as the steepest ascent flow for the area functional of spacelike hypersurfaces. The analogue of equation (4) with $\mathcal{H} = 0$ in Riemannian manifolds is the so-called mean curvature flow which has been the focus of an extensive study for more than a decade (see [H1]). The flow for nonzero $\mathcal{H}$ is not well behaved there in terms of convergence to stationary solutions even in the case of constant $\mathcal{H}$. However in Lorentzian manifolds this flow is very robust as we will see below.

Equation (4) satisfies parabolic maximum and comparison principles. We state them next as they will greatly facilitate the discussion of the qualitative behaviour of solutions of (4) in comparison to some explicit examples given below.

Comparison Principle. *Let u_1 and u_2 be solutions of (4) on a bounded domain $\Omega \subset \mathbf{R}^n$. Suppose that $u_1(x,0) \leq u_2(x,0)$ for all $x \in \Omega$ and $u_1(x,t) \leq u_2(x,t)$ for all $x \in \partial\Omega$ and $t \geq 0$. Then $u_1(x,t) \leq u_2(x,t)$ for all $x \in \Omega$ and $t \geq 0$.*

Maximum Principle. *Let $M_t = \operatorname{graph} u(\cdot,t)$ where $u : \Omega \times [0,T) \to \mathbf{R}$ solves (4) for $t \in (0,T)$. Suppose the function $f : \Omega \times [0,T) \to \mathbf{R}$ satisfies the equation*

$$\left(\frac{d}{dt} - \Delta\right) f = \langle a, \nabla f\rangle + bf$$

for $t \in (0,T)$ where ∇ and Δ denote the tangential gradient and the Laplace-Beltrami operator on M_t respectively and where we assume that $a : \Omega \times [0,T) \to \mathbf{R}^{n,1}$ and $b : \Omega \times [0,T) \to \mathbf{R}$ are bounded. If $\operatorname{spt} f(\cdot,t)$ is compact for every $t \in [0,T)$ then

$$f(\cdot,0) \leq 0 \;\Rightarrow\; f(\cdot,t) \leq 0 \;\forall t \in [0,T].$$

If

$$\left(\frac{d}{dt} - \Delta\right) f \leq \langle a, \nabla f\rangle$$

then

$$\sup_{M_t} f \leq \sup_{M_0} f.$$

The comparison and the maximum principle continue to hold for the flow of noncompact hypersurfaces as long as the hypersurfaces M_t have bounded second fundamental form and the function f evolving on them satisfies certain conditions regarding its growth at infinity (see [EH2]).

3. Flow with $\mathcal{H} = 0$

Homothetic solutions. The simplest nontrivial example consists of the spacelike hyperboloids of constant mean curvature $n/\sqrt{\rho^2 + 2nt}$ given by the functions

$$u_\rho(x,t) = \sqrt{|x-y|^2 + \rho^2 + 2nt}$$

where $y \in \mathbf{R}^n$ is fixed. They are homothetic solutions with initial data given by the constant mean curvature hyperboloids u_ρ. For $\rho = 0$ the initial hypersurface is the upper light cone at $(y, 0)$. These solutions remain asymptotic to their initial data for all $t > 0$ that is do not move at null infinity. Comparison of a general solution to these homothetic ones is used to obtain height estimates. As long as an entire solution of (4) is controlled at infinity we can use the comparison principle to infer that it stays underneath a homothetic solution if it does so initially. We can conclude the same from the maximum principle:

Note first that the Lorentz distance function $z = \langle X, X \rangle$ satisfies the evolution equation

$$\left(\frac{d}{dt} - \Delta\right)(z + 2nt) = 0.$$

This can be seen directly from (3') or by calculating $\frac{d}{dt}z = 2H\langle X, \nu\rangle$ and using $\Delta z = 2(n + H\langle X, \nu\rangle)$ (see for example [BS]). Assume now that M_0 lies initially underneath a hyperboloid given by u_ρ. This says $z > -\rho^2$ at time $t = 0$. The maximum principle applied to z then yields $z > -\rho^2 - 2nt$ on M_t which translates into the statement

$$u_0^2(x) < |x|^2 + \rho^2 \Rightarrow u^2(x,t) < |x|^2 + \rho^2 + 2nt.$$

By comparing a solution of (4) to the homothetic solutions flowing out of light cones at every point on M_0 we obtain the following height estimate (see [E2]) which we state only in the compact case:

Height Estimate. *Let Ω be a bounded domain in $\mathbf{R}^n$ and $u_0 : \Omega \to \mathbf{R}$ be spacelike. Let u be a solution of (4) with $\mathcal{H} = 0$ in $\Omega \times (0,T)$ which satisfies $u(\cdot, 0) = u_0$ in Ω and $u(\cdot, t) = u_0$ on $\partial\Omega$ for $t \geq 0$. Then for all $x \in \Omega$ and $t \in [0,T]$ we have the inequality*

$$|u(x,t) - u_0(x)| \leq \sqrt{2nt}.$$

For the proof we notice that since u_0 is spacelike the inequality

$$u_0(y) - |x-y| < u_0(x) < u_0(y) + |x-y|$$

holds for all $x, y \in \Omega$. For every $y \in \Omega$ we use the comparison principle to control the solution u by the homothetic solutions given by

$$u_0(y) \pm \sqrt{|x-y|^2 + 2nt}.$$

These solutions have initial data $u_0(y) \pm |x-y|$. Therefore the inequality

$$u_0(y) - \sqrt{|x-y|^2 + 2nt} \leq u(x,t) \leq u_0(y) + \sqrt{|x-y|^2 + 2nt}$$

holds for every $x \in \Omega$. Setting $x = y$ implies the estimate.

Translating solutions. Height estimates such as the above can no longer be expected for entire solutions unless we impose restrictions on their behaviour at infinity. In fact, there are solutions of (4) moving by vertical translation which provide a counterexample: Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be an initial spacelike hypersurface which satisfies the elliptic equation (2) above. Then u defined by

$$u(x,t) = u_0(x) + t$$

solves equation (4). Note in particular that the comparison principle does not apply in this case. In fact, the translating solution given by $u(x,t) = \log\cosh x + t$ lies initially underneath the homothetic solution given by $\sqrt{x^2 + 2t}$ but crosses it at infinity at time $t = \log 2$.

We are currently investigating the longterm behaviour of solutions of (4) with $\mathcal{H} = 0$ in asymptotically flat spacetimes with particular focus on trying to find geodesically incomplete solutions which for $t \to \infty$ exhibit similar behaviour to the translating solutions introduced here. Note that such solutions suggest a canonical choice of compact coordinates on a spacetime.

4. Flow for general $\mathcal{H}$.

In the following discussion we will always consider a class of entire solutions to which the noncompact maximum and comparison principles apply. For a more precise discussion of sufficient conditions in some special situations we refer to [E1].

We particularly would like to keep the cases $\mathcal{H} = 0, \mathcal{H} = \frac{n}{\rho}$ and $\mathcal{H} = v$ in mind. The flow with $\mathcal{H} = 0$ does not move spacelike hyperplanes, in fact by the Cheng and Yau result ([CY]) stated above these are the only stationary solutions of this flow. The n/ρ - flow preserves the spacelike hyperboloids and the v - flow does not affect solutions of the elliptic equation (2). We therefore expect a general solution of (4) with appropriate restrictions on their initial data to behave like the stationary solutions for $t \to \infty$. They may not always converge to the stationary solution as is exemplified by the homothetic solutions of the $\mathcal{H} = 0$ - flow which flatten out since their mean curvature satisfies

$$H(t) = \frac{n}{\sqrt{\rho^2 + 2nt}} \to 0$$

as $t \to \infty$ but where the solutions become unbounded. The translating example does not even become flat but due to its exponential curvature growth this is outside the class of solutions we consider anyway.

To obtain a better understanding let us consider the behaviour of the quantities v and $H - \mathcal{H}$ on the evolving hypersurfaces M_t. In [EH1], the evolution equations

$$(5) \qquad \left(\frac{d}{dt} - \Delta\right) v = -v|A|^2 + \langle e_0, \nabla \mathcal{H}\rangle$$

$$(6) \qquad \left(\frac{d}{dt} - \Delta\right) (H - \mathcal{H}) = -(H - \mathcal{H})(|A|^2 + \langle D\mathcal{H}, \nu\rangle)$$

were derived, where D denotes differentiation in $\mathbf{R}^{n,1}$ and $|A|^2$ is the squared norm of the second fundamental form of M_t.

Let us discuss the case of constant $\mathcal{H}$. For the purpose of this talk, the additional assumption that our initial hypersurface satisfies $H \geq \mathcal{H}$ will help us keep technicalities to a minimum. A more general result without this condition on M_0 was proved by Stavrou [St] in the case $\mathcal{H} = 0$ and generalized to flow in asymptotically flat spacetimes in [E1]. In the case of cosmological spacetimes where we evolve compact Cauchy surfaces analogous results were proved in [EH1] for more general $\mathcal{H}$ than the constant one considered here.

Proposition. *Let $\mathcal{H} \equiv$ const. Suppose that M_0 lies underneath an entire spacelike hypersurface M^+ satisfying $H_{M^+} \leq \mathcal{H}$. Suppose furthermore that $\sup_{M_0} v < \infty$ and $\sup_{M_0} |A| < \infty$ and that $H_{M_0} \geq \mathcal{H}$. Then (3) has a solution M_t which will move towards M^+ and which for $t \to \infty$ converges to a hypersurface M_∞ satisfying $H = \mathcal{H}$. In the case of nonzero $\mathcal{H}$ this convergence occurs at an exponential rate.*

In particular, for $\mathcal{H} = \frac{n}{\rho}$ an initial hypersurface M_0 which lies underneath the hyperboloid u_ρ and satisfies $H \geq \frac{n}{\rho}$ will move towards u_ρ and converge to a spacelike hypersurface with constant mean curvature $\frac{n}{\rho}$.

Proof of Proposition. The condition $\sup_{M_0} |A| < \infty$ guarantees that (3) has a smooth solution for a short time interval. Since $\mathcal{H}$ is constant equations (5) and (6) have the form required for the maximum principle. We can therefore immediately conclude that

$$\sup_{M_t} v \leq \sup_{M_0} v$$

and that $H \geq \mathcal{H}$ on M_t for $t > 0$. Moreover, one easily calculates from (5) using the inequality $|A|^2 \geq \frac{1}{n}H^2$ that $H - \mathcal{H}$ satisfies

$$(7) \qquad \left(\frac{d}{dt} - \Delta\right) (H - \mathcal{H})^2 \leq -\frac{2}{n}H^2(H - \mathcal{H})^2.$$

The estimate on v guarantees that equation (4) is uniformly parabolic for all times. Therefore, the solution stays smooth (which follows from standard parabolic theory or from direct curvature estimates in [EH1] and [E1] using the initial bound on $|A|$). If $|\mathcal{H}| = \mathcal{H}_0 > 0$ we have $H^2 \geq \mathcal{H}_0^2$ and therefore obtain from (7) that

$$\left(\frac{d}{dt} - \Delta\right) e^{\frac{2}{n}\mathcal{H}_0^2 t}\,(H - \mathcal{H})^2 \leq 0.$$

Hence by the maximum principle $(H - \mathcal{H}) \to 0$ at an exponential rate. Since $H - \mathcal{H}$ is the speed for the height function u of M_t this implies exponential convergence of M_t to some spacelike hypersurface satisfying $H = \mathcal{H}$. When $\mathcal{H} = 0$ we cannot use this argument but observe that equation (4) actually says that

$$\frac{\partial}{\partial t} u = (H - \mathcal{H})\, v^{-1}.$$

We now integrate this equation with respect to time and observe that due to the nonnegativity of $H - \mathcal{H}$ and the comparison principle (which also holds for super - and subsolutions of (4) such as M^+) we have the inequality

$$u^+(x) \geq u(x,t) \geq u_0(x)$$

for all $x \in \mathbf{R}^n$ and $t \geq 0$ where $M^+ = \operatorname{graph} u^+$. Convergence then follows in view of the uniform estimates and the fact that the solution always moves in the same direction. In fact, we see that $H - \mathcal{H}$ has to converge to zero.

For more general $\mathcal{H}$ the situation is not as straightforward. We first need to estimate the terms involving the gradients of $\mathcal{H}$. If $\mathcal{H} \in C^1(\mathbf{R}^{n,1})$ we estimate $|\langle D\mathcal{H}, \nu\rangle| \leq |D\mathcal{H}|v$ and $|\langle e_0, \nabla\mathcal{H}| \leq |D\mathcal{H}|v$. If $\mathcal{H}$ also depends on future directed timelike vectorfields these terms will be even more unpleasant as they will then also depend on the norm of the second fundamental form $|A|$. For example for $\mathcal{H} = v$ we have $|\langle D\mathcal{H}, \nu\rangle| \leq |A|v^2$ and $|\langle e_0, \nabla\mathcal{H}\rangle| \leq |A|v^2$.

In [B1], Bartnik studied the elliptic equation (1) for hypersurfaces of prescribed mean curvature $\mathcal{H}$. He obtained a priori estimates for v when $\mathcal{H}$ satisfied certain structure conditions. The case $\mathcal{H} = v$ is included in his analysis. An important step in his work which features also in other gradient estimate arguments (see [B1,2], [CY], [G]) is the use of the inequality

$$|A|^2 v \geq (1 + \frac{1}{n})|\nabla v|^2 v^{-1} - H^2 v. \tag{8}$$

which holds for a general spacelike hypersurface. Let me indicate how one proceeds with Bartnik's argument in the flow case. If we also estimate

$$|A|v^2 \leq \epsilon |A|^2 v + \frac{1}{4\epsilon} v^3$$

and choose $\epsilon = \epsilon(n) > 0$ appropriately then (5) and (8) imply

$$\left(\frac{d}{dt} - \Delta\right) v \leq -\left(1 + \frac{1}{2n}\right) |\nabla v|^2 v^{-1} + c(n)(H^2 v + v^3). \tag{9}$$

In Bartnik's case we only have the Laplace operator and the H^2 - term is bounded whenever $\mathcal{H}$ is. The v^3 - term can be controlled by calculating the Laplacian of the function $e^{\lambda u} v$ where $u = -\langle X, e_0 \rangle$ noting that $\Delta u = \mathcal{H} v$ in view of equation (1). For sufficiently large $\lambda > 0$ depending on a bound of u (this only works for bounded solutions) the $|\nabla v|$ - term can then be used to dominate the expression involving v^3. For details see [B1].

In the case of the flow equation (4) the H^2 - term has to be estimated first before Bartnik's argument can be adapted. If we assume that $\langle D\mathcal{H}, \nu \rangle$ is nonnegative (see [EH1], this follows for example if $\mathcal{H}$ decreases into the future) then equation (5) (or inequality (7)) and the maximum principle can be used to derive a bound on $H - \mathcal{H}$ which yields an a priori estimate for H. We are then in the same situation as in the elliptic case and can proceed to bound v as long as we have a priori control on the height function u. In the case of entire solutions, u is generally not bounded so some interior estimate arguments are required which we will describe later. However, in asymptotically flat spacetimes this process applied to bounded solutions can be used to flow to noncompact spacelike hypersurfaces which are asymptotic to a given reference time slice at spatial infinity (see [E1]).

If we do not assume a sign condition on $\langle D\mathcal{H}, \nu \rangle$ the situation becomes more complicated. In [E1], the case of a general $\mathcal{H} = \mathcal{H}(\cdot, u)$ was treated. The essential idea there was to suitably combine the evolution equations (5) and (6) above so that v and $H - \mathcal{H}$ can be estimated simultaneously.

The case where $\mathcal{H}$ depends on u and Du is still open. In the case $\mathcal{H} = v$ one notices that $\langle D\mathcal{H}, \nu \rangle \geq 0$ on convex spacelike hypersurfaces. One could check whether equation (4) preserves convexity of an initial hypersurface and then work in this class. If we then started initially underneath a hypersurface M^+ satisfying $H \leq v$ and our initial hypersurface M_0 apart from being convex satisfied $H \geq v$, the flow equation (4) with $\mathcal{H} = v$ should drive the solution M_t towards a hypersurface satisfying $H = v$.

5. Interior estimates.

In the previous section we made the major assumption that our solution hypersurfaces satisfy conditions at infinity which ensure that the noncompact maximum principle is applicable. As we have seen, the maximum and comparison principle are not applicable to some of the most interesting solutions such as the translating ones. Nevertheless we are able to prove that equation (4) in the case $\mathcal{H} = 0$ always has a global smooth solution without making any assumptions on the initial data (see [E2]). In particular, we are allowed to start with geodesically incomplete hypersurfaces:

Theorem. *Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be spacelike and smooth. Then the equation*

$$\frac{\partial u}{\partial t} = \sqrt{1-|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)$$

has a smooth solution for all $t > 0$ with initial data u_0. Moreover, this solution satisfies the a priori height estimate

$$|u(x,t) - u_0(x)| \leq \sqrt{2nt}.$$

A nonuniqueness example. An interesting consequence of the height estimate for the particular solution constructed in te theorem is that it implies that in general solutions are not unique. Consider for example the initial data given by $u_0(x) = \log\cosh x$. These give rise to the translating solution $\log\cosh x + t$ which has distance t from its initial data. By the theorem there is another solution which only has distance $\sqrt{2nt}$ from its initial data and therefore has to be different from the translating one.

The proof of the above theorem relies on the following new interior estimate ([E2]) which simultaneously controls the gradient function v and the mean curvature H of M_t inside the set

$$K_R(0) = \left\{X \in \mathbf{L}^{n+1},\, z = \langle X, X\rangle \leq R^2\right\}.$$

We will always assume that the hypersurfaces M_t have no boundary inside the sets in which we are estimating.

Proposition. *Suppose that $M_t \cap K_R(0)$ is compact in $\mathbf{R}^{n,1}$ for $t \in [0, \frac{R^2}{2n}]$. Let $\Lambda > \sup_{M_0 \cap K_R(0)} H^2$. There are constants $p, q > 0$ which only depend on n such that for all $t \in [0, \frac{R^2}{2n}]$*

$$\sup_{M_t}\left(v^2 \frac{1}{(\Lambda - H^2)^{1/q}}(R^2 - z - 2nt)^p\right) \leq e^{c(n)q\Lambda t} \sup_{M_0}\left(v^2 \frac{1}{(\Lambda - H^2)^{1/q}}(R^2 - z)^p\right).$$

This estimate can be thought of as a parabolic analogue of a local estimate for v for solutions of the maximal surface equation (see [B3]). One can show that this interior estimate still holds in the case of a general forcing term $\mathcal{H} = \mathcal{H}(\cdot, u)$ and that the existence result carries over to equation (4) for this case.

The proposition implies that

$$\sup_{M_t \cap K_R(0)} (v + |H|) \leq c_1$$

for $t \leq c(n)R^2$ where c_1 depends on n, R and $\sup_{M_0 \cap K_{2R}(0)}(v + |H|)$. We can then proceed to derive similar bounds for the second fundamental form of M_t and its covariant derivatives, see [E2].

To prove existence of an entire solution of (4) we also need the following result ensuring the existence of a solution to the initial-boundary value problem on bounded domains in $\mathbf{R}^n$ ([E2]):

Proposition. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary. Let $u_0 : \bar{\Omega} \to \mathbf{R}$ be smooth and strictly spacelike in the sense that $\sup_{\bar{\Omega}} |Du_0| < 1$. Then the equation*

$$\frac{\partial u}{\partial t} = \sqrt{1 - |Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right)$$

has a smooth solution in Ω for all $t > 0$ which satisfies $u(\cdot, 0) = u_0$ in Ω and $u(\cdot, t) = u_0$ on $\partial\Omega$. Moreover, as $t \to \infty$, $u(\cdot, t)$ converges smoothly to the unique solution of the maximal surface equation with boundary data u_0.

The proof of this proposition is a straightforward adaptation of arguments in [BS] and [H2].

The existence theorem is now proved in the following way:

Suppose without loss of generality that $u_0(0) = 0$. For $k \in \mathbf{N}$, we let u_k be the smooth solution of the initial-boundary value problem

$$\begin{aligned} \frac{\partial u_k}{\partial t} &= \sqrt{1 - |Du_k|^2}\,\mathrm{div}\left(\frac{Du_k}{\sqrt{1 - |Du_k|^2}}\right) && \text{in} \quad B_k(0) \times (0, \infty) \\ u_k(\cdot, 0) &= u_0 && \text{in} \quad B_k(0) \\ u_k(\cdot, t) &= u_0 && \text{on} \quad \partial B_k(0) \times (0, \infty). \end{aligned}$$

Fix $R > 0$. Since u_0 is spacelike and $u_0(0) = 0$ we have that

$$|x|^2 - u_0^2(x) \to \infty$$

as $|x| \to \infty$. Hence for sufficiently large k depending on R we have that $|x|^2 - u_0^2(x) > 16R^2$ for $|x| = k$. For $M_t^k = \mathrm{graph}\, u_k(\cdot, t)$ this implies that $\partial M_t^k \cap K_{4R}(0) = \emptyset$ for all $t \geq 0$. Also, $M_t^k \cap K_{4R}(0)$ is compact for $t \geq 0$ as these sets are contained in the cylinders $B_k(0) \times \mathbf{R}$. We can therefore apply the interior estimates stated above to the solution (M_t^k) inside $K_{4R}(0)$ to obtain for $t \leq c(n)R^2$

$$\sup_{M_t^k \cap C_{2R}(0)} (v + |H|) \leq c_1.$$

We also employ the uniform estimates for the second fundamental form and its covariant derivatives (see [E2]) to obtain

$$\sup_{M_t^k \cap C_R(0)} |\nabla^m A|^2 \leq c_m$$

for $t \in [0, c(n)R^2]$ and for all $m \geq 0$. These estimates translate into uniform bounds (independent of k) on $B_R(0) \times [0, c(n)R^2]$ for $v(u_k)$ and derivatives of all orders of u_k. The height estimate yields for every k that

$$|u_k(x, t) - u_0(x)| \leq \sqrt{2nt} \tag{10}$$

for all $x \in B_k(0)$ and $t \geq 0$ This implies the estimate

$$\sup_{B_R(0)\times[0,c(n)R^2]} |u_k| \leq c(n, R, \sup_{B_R(0)} |u_0|)$$

which is independent of k.

Since R is arbitrary we can now select a subsequence of (u_k) for $k \to \infty$ which converges smoothly on compact subsets of $\mathbf{R}^n \times [0, \infty)$ to a solution u of (4). In view of the uniform convergence the desired height estimate follows from (10).

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