STRUCTUAL INEQUALITIES METHOD FOR UNIQUENESS THEOREMS FOR THE MINIMAL SURFACE EQUATION

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1. Existence theorems for elliptic equations

Consider the second order elliptic equations in divergence form:

$$Qu = \mathrm{div}A(x,u,Du) + B(x,u,Du) = 0$$

where $Du = \langle D_1 u, ..., D_n u\rangle = \langle \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\rangle$, $A = \langle A_1(x,u,Du), ..., A_n(x,u,Du)\rangle$ and $(A(x,u,Du) - A(x,v,Dv)) \cdot (Du - Dv) > 0$ for every $u, v \in C^1$, $Du \neq Dv$. And consider the Dirichlet problem

$$(1) \qquad \begin{cases} Qu = \mathrm{div}A(x,u,Du) + B(x,u,Du) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

It is well-known that the solvability of (1) depends on the structural conditions A, B and the geometric properties of Ω. For example, consider the operator $Qu = \partial_i(a^{ij}u_j) + B$ which is uniformly elliptic. Then the Dirichlet problem (1) is solvable for any bounded smooth domain with continuous boundary value φ.

Now, consider the minimal surface equation $\mathrm{div}Tu = 0$ in Ω where Ω is a bounded smooth domain. Then the Dirichlet problem $\mathrm{div}Tu = 0$ in $\Omega, u = \varphi$ on $\partial\Omega$ is solvable for arbitrary continuous boundary value φ if and only if the mean curvature of $\partial\Omega$ is everywhere non-negative [7].

It this talk, by using some structural inequalities for divTu, we will establish some uniqueness theorems for the minimal surface equation and prescribed mean curvature equations.

2. Removability singularity for the minimal surface equation

It is well-known that $\Delta \log r = 0$ in $\mathbb{R}^2\backslash\{0\}$ where $r = \sqrt{x^2+y^2}$ and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In 1951, Bers [1] anounced that the solution for the minimal surface equation has no isolated singularity.

Theorem 1. *Let $u \in C^0(\bar{B}_R\backslash\{0\}) \cap C^2(B_R\backslash\{0\})$ with div$Tu = 0$ in $B_R\backslash\{0\}$ where $B_R \subset \mathbb{R}^n$ is the open ball of radius $R > 0$. Then $u \in C^2(B_R)$.*

Based on Concus-Finn's device [2], [3], a simpler proof of Theorme 1 can be obtained as follows. We begin with the following theorem:

Theorem 2. *Let $u, v \in C^0(\bar{B}_R\backslash\{0\}) \cap C^2(B_R\backslash\{0\})$ and suppose that divTu = divTv in $B_R\backslash\{0\}$, $u = v$ on ∂B_R. Then $u \equiv v$ in $B_R\backslash\{0\}$.*

Proof. Let $0 < \varepsilon < R$, then

$$\int_{\partial(B_R\backslash B_\varepsilon)} \tan^{-1}(u-v)(Tu-Tv)\cdot\nu = \int_{B_R\backslash B_\varepsilon} \frac{(Du-Dv)}{1+(u-v)^2}\cdot(Tu-Tv) \geq 0$$

where ν is the unit outward normal to $\partial(B_R\backslash B_\varepsilon)$. Since $u-v=0$ on ∂B_R, $|\tan^{-1}(u-v)| \leq \frac{\pi}{2}$, $|Tu-Tv| \leq 2$, we have $|\int_{\partial(B_R\backslash B_\varepsilon)} \tan^{-1}(u-v)(Tu-Tv)\cdot\nu| \leq |\partial B_\varepsilon|\cdot\pi$ where $|\partial B_\varepsilon|$ is the $(n-1)$-dimensional Hausdorff measure for ∂B_ε.

Let $\varepsilon \to 0$, we have $\iint_{B_R\backslash\{0\}} \frac{(Du-Dv)\cdot(Tu-Tv)}{1+(u-v)^2} = 0$. Since $(Du-Dv)\cdot(Tu-Tv) > 0$ if and only if $Du \neq Dv$, we have $Du \equiv Dv$ in Ω. The theorem is proved.

Concus-Finn's method is based on the following structural inequality

$$|Tu| \leq \text{ bounded} \tag{2}$$

only. In fact, if $\{0\}$ is replaced by a set S with $(n-1)$-dimensional Hausdorff measure zero, it is easy to see that Theorem 2 is true also.

Proof of Theorem 1. Consider the Dirichlet problem $\text{div}Tw = 0$ in B_R, $w = u$ on ∂B_R. Since B_R is convex, there exists a solution $w \in C^2(B_R) \cap C^0(\bar{B}_R)$ of the above Dirichlet problem. By Theorem 2, we have $u \equiv w \in C^2$ and the theorem follows.

3. Uniqueness theorems for capillary surfaces

Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain and let $\partial\Omega \in C^1$, consider the capillary surface equations in a uniform gravitational field:

$$\begin{cases} \text{div}Tu = k_0 u & \text{in } \Omega \\ Tu \cdot \nu = \cos\gamma & \text{on } \partial\Omega \end{cases} \tag{3}$$

where k_0 is a positive constant, $-\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}$.

Theorem 3. *(Finn-Hwang [6], Kurta [11]) If (3) has a solution, then it is unique.*

The proof of Theorem 3 is based on the following structural inequality

$$|Tu| \leq \quad \text{bounded} \tag{2}$$

also.

For the capillary surface equations without gravity, Tam [14], [15] proved the following theorem:

Theorem 4. *Let $\Omega = (a,b) \times \mathbb{R}$ be an infinite strip in $\mathbb{R}^2$ where $a < b$ are constants. Let $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ and suppose that*

$$\begin{cases} \mathit{div}Tu = H_0 & \mathit{in}\ \Omega \\ Tu \cdot \nu = \cos\gamma_0 & \mathit{on}\ \partial\Omega \end{cases}$$

where H_0, γ_0 are constants, $0 \le \gamma_0 \le \frac{\pi}{2}$ and $H_0 = 2\cos\gamma_0$. Then u must be a cylinder.

By the following structural inequality

$$(4) \qquad (Du - Dv)\cdot(Tu - Tv) \ge \frac{|Du - Dv|^2}{\sqrt{1 + (|Du| + |Du - Dv|)^2}}\left(1 - \frac{|Du|}{\sqrt{1 + |Du|^2}}\right),$$

Hwnag [9] gived a simpler proof for Theorem 4 and generalize it.

4. Nitsche's conjecture

In 1965, Nitsche [13] announced the following theorem:

Theorem 5. *Let $\Omega_\alpha \subset \mathbb{R}^2$ be a sector with angle $0 < \alpha < \pi$ and let $\Omega \subset \Omega_\alpha$. Suppose that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and $divTu = 0$ in Ω, $u \le ax + by + c$ on $\partial\Omega$, where a, b, c are constants. Then $u \le ax + by + c$ in Ω.*

Hence Nitsche raised the following conjecture: "Let $\Omega \subset \Omega_\alpha$ be an unbounded domain where Ω_α be as above. Let $\varphi \in C^0(\partial\Omega)$ and suppose that the Dirichlet problem $\text{div}Tu = 0$ in Ω, $u = \varphi$ on $\partial\Omega$ has a solution, is it unique?" Results in this direction were obtained by Miklyukov [12] and Hwang [8] independently in the following theorem:

Theorem 6. *Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain and let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$. For every $R > 0$, set $\Gamma_R = \partial(\Omega \cap B_R) \cap \partial B_R$. Suppose that*

$$\begin{cases} (i) \ \ divTu = divTv & \text{in } \Omega \\ (ii) \ \ u = v & \text{on } \partial\Omega \\ (iii) \ \ \max_{\Omega \cap B_R}(u - v) = O\left(\sqrt{\int_{R_0}^{R} \frac{1}{|\Gamma_r|} dr}\right) & \text{as } R \to \infty \end{cases}$$

for some positive constants R_0. Then $u \equiv v$ in Ω.

A stronger version of Theorem 6 was discovered by Collin-Krust [5] independently, which is the following:

Theorem 7. *Let $\Omega, u, v, \Gamma_r, |\Gamma_r|$ be as in Theorem 6. Suppose that*

$$\begin{cases} (i)\ \ divTu = divTv & in\ \ \Omega \\ (ii)\ \ u = v & on\ \ \partial\Omega \\ (iii)\ \ \max_{\Omega \cap B_R} |u - v| = o(\int_{R_0}^{R} \frac{1}{|\Gamma_r|} dr) & as\ \ R \to \infty \end{cases}$$

for some positive constant R_0. Then $u \equiv v$ in Ω.

In fact, for any unbounded domain $\Omega \subset \mathbb{R}^2$, we have $|\Gamma_R| \leq 2\pi R$, and condition (iii) in Theorem 7 becomes $\max_{\Omega \cap B_R} |u - v| = o(\log R)$ as $R \to \infty$.

In the special case when Ω is a strip, then $|\Gamma_R| \leq$ constant and condition (iii) becomes $\max_{\Omega \cap B_R} |u - v| = o(R)$ as $R \to \infty$. On the other hand, in a strip domain Ω, Collin [4] showed that there exist two solutions for the minimal surface equation with $u = v$ on $\partial\Omega$ and $\max_{\Omega \cap B_R} |u - v| = O(R)$ as $R \to \infty$. So condition (iii) in Theorem 7 is necessary.

This counterexample also answers Nitsche's conjecture in the negative. In contrast, the following result is also given by Collin-Krust [5]:

Theorem 8. *Let $\Omega = (0, 1) \times \mathbb{R}^2$ be a strip and let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Suppose that $divTu = 0$ in $\Omega, u(0, y) = ay + b, u(1, y) = cy + d$, where a, b, c, d are constants. Then u must be a helicoid.*

The following inequality was discovered by Miklyukov [12], Hwang [8] and Collin-Krust [5]:

$$(Tu - Tv) \cdot (Du - Dv) \geq \frac{\sqrt{1 + |Du|^2} + \sqrt{1 + |Dv|^2}}{2} |Tu - Tv|^2 \tag{5}$$
$$\geq |Tu - Tv|^2.$$

Using this inequality, Miklyukov and Hwang proved Theorem 6 independently and Collin-Krust proved Theorem 7 also based on (5).

It seems that (5) can not be used to prove Theorem 8, and so Collin-Krust resorted the theory of Gauss maps instead. But Hwang [10] pointed out that (5) could be used to give a simpler proof of Theorem 8.

As a conclusion, we would like to emphasize that since Tu satisties different structual inequalities (2), (4) and (5), we can obtain different uniqueness theorems. We are interested to know whether we can apply the same method to different new structural inequalities and obtain new uniqueness theorems.

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