Cellular Neural Networks: Pattern Formation and Spatial Chaos

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Abstract

We consider a Cellular Neural Network (CNN) with a bias term z in the integer lattice $\mathbf{Z}^2$ on the plane $\mathbf{R}^2$. We impose a symmetric coupling between nearest neighbors, and also between next-nearest neighbors. Two parameters, a and ε, are used to describe the weights between such interacting cells. We study patterns that can exist as stable equilibria. In particular, the relationship between mosaic patterns, and the parameter space $(z, a; \varepsilon)$ can be completely characterized. This, in turn, addresses the so-called "Learning Problem" in CNNs. The complexities of mosaic is also studied.

I. Introduction

In this talk, we study Cellular Neural Networks (CNNs) without input terms, and of the form

$$\frac{dx_{i,j}}{dt} = -x_{i,j} + z + \sum_{|k|\leq 1, |\ell|\leq 1} a_{k,\ell} f(x_{i+k,j+\ell}),\ \ (i,j) \in \mathcal{Z}^2, \tag{1.1a}$$

$$x_{i,j}(0) = x^0_{i,j}. \tag{1.1b}$$

Here the nonlinearity f is a piecewise-linear function of the form

$$f(x) = \frac{1}{2}(|x+1| - |x-1|). \tag{1.2}$$

The numbers $a_{k,\ell}$, $|k| \leq 1$, $|\ell| \leq 1$, $k, \ell \in \mathcal{Z}$, are arranged in a 3×3 matrix form, which is called a space-invariant A-template

$$A = \begin{bmatrix} a_{-1,1} & a_{01} & a_{1,1} \\ a_{-1,0} & a_{0,0} & a_{1,0} \\ a_{-1,-1} & a_{0,-1} & a_{1,-1} \end{bmatrix}. \tag{1.3}$$

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The quantities $x_{i,j}$ denote the state of a cell $C_{i,j}$. If $x_{i,j} > 1$ (resp., $x_{i,j} < -1$, then its corresponding cell $C_{i,j}$ is called a positively (resp., negatively) saturated cell. If $|x_{i,j}| < 1$, then its associated cell $C_{i,j}$ is called a defect cell or a defect. The output of a cell $C_{i,j}$, defined as $y_{i,j} = f(x_{i,j})$, and is thus always bounded by $|y_{i,j}| \leq 1$. The quantity z is an independent voltage source. When $z = 0$, (1.1) is called unbiased, and is called biased when $z \neq 0$.

CNN systems were first proposed by Chua and Yang in [5,6]. Such systems share the best features of neural networks and cellular automata, their continuous-time feature allows real-time signal processing absent from the digital domain, and their local interconnection feature makes them ideal for VLSI implementation. Moreover, Chua constructed an electrical circuit on a chip that simulates a CNN system. For additional background information, applications and theory, see [4, 5, 6, 7, 8] among others.

Lattices also play important and in some cases essential roles in many scientific models, typically modeling underlying spatial structures. We mention in particular, models arising from chemical reactions, biology, material science, and image-processing and pattern-recognition. Much theoretical work in lattice differential equations concerns one-dimensional lattices. Some theoretical approaches to systems of higher dimensions have been made; see e.g., [1, 2, 3].

Stationary solutions $\bar{x} = (\bar{x}_{i,j})$ of (1.1a) are important in studying CNN systems; their outputs $\bar{y} = (f(\bar{x}_{i,j}))$ are called patterns. Two types of stationary solution are of interest: mosaic and defect. A mosaic solution $\bar{x}$ satisfies $|\bar{x}_{i,j}| > 1$ for all $(i,j) \in \mathcal{Z}^2$. A defect solution $\bar{x}$ satisfies $|\bar{x}_{i,j}| > 1$ for $(i,j) \in \mathcal{Z}^2 \backslash D$ and $|\bar{x}_{k,\ell}| < 1$ for $(k,\ell) \in D$, where $D \neq \phi$ and $D \neq \mathcal{Z}^2$. Their corresponding pattern $\bar{y}$ can thus be called a mosaic and a defect pattern, respectively. It is known the mosaic solution are necessary stable.

One basic problem in CNN theory is the so-called "Learning Problem", which can be stated as follows:

(i) Given a set of stationary patterns $\mathcal{U}$, determine a set of parameters $\mathcal{P} \subset \mathcal{P}_{10} = \{z, a_{k,\ell} : k, \ell \text{ integer and } |k|, |\ell| \leq 1\}$, and a parameter space, such that any pattern in $\mathcal{U}$ can be obtained and is stable for all parameters in $\mathcal{P}$. (1.4a)

The "Learning Problem" (i) is almost the inverse of the following problem.

(ii) Given any $\mathcal{P} \subset \mathcal{P}_{10}$, determine $\mathcal{M}(\mathcal{P})$ (resp., $\mathcal{D}(\mathcal{P})$), the set of all stable mosaic (resp., defect) patterns of (1.1). (1.4b)

Furthermore, we also wish to address

(iii) the complexity of $\mathcal{M}(\mathcal{P})$ and $\mathcal{D}(\mathcal{P})$ for each subset $\mathcal{P}$ of $\mathcal{P}_{10}$. (1.4c)

To study these problems, we begin with a local solution x_T of (1.1a) for a certain subsets T of $\mathcal{Z}^2$. We find that the parameter space $\mathcal{P}_{10}$ can be partitioned into finitely many regions $\{\mathcal{P}^{(k)}\}_{k\in K}$. Only a few local patterns are allowed in each region $\mathcal{P}^{(k)}$, these are called the feasible patterns of region $\mathcal{P}^{(k)}$. In principle, we can obtain all stable patterns by patching these feasible patterns together. However, to construct all stable patterns of $\mathcal{P}^{(k)}$ more efficiently, we introduce a set $B(\mathcal{P}^{(k)})$ of "building blocks" for each region $\mathcal{P}^{(k)}$. Then, using certain compatibility rules $\mathcal{C}(\mathcal{P}^{(k)})$, we can patch these building blocks together into a global pattern in $\mathcal{Z}^2$. These building blocks and compatibility conditions also enable us to estimate the spatial entropy of $\mathcal{M}(\mathcal{P}^{(k)})$ and $\mathcal{D}(\mathcal{P}^{(k)})$, the set of all mosaic patterns and defect patterns, respectively.

For simplicity, in this talk we emphasis the case in which template A is a square cross, e.g.,

$$A = \begin{bmatrix} 0 & b & 0 \\ b & a & b \\ 0 & b & 0 \end{bmatrix}. \tag{1.5}$$

For this case, we completely solve the problems in (1.4) for the set of stable mosaic patterns. The method is quite general and can be applied to more general templates A.

2. Partitioning the Parameter Spaces

Let template A be square-crossed; e.g.,

$$A = A^+ \equiv \begin{bmatrix} 0 & b & 0 \\ b & a & b \\ 0 & b & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & a\varepsilon & 0 \\ a\varepsilon & a & a\varepsilon \\ 0 & a\varepsilon & 0 \end{bmatrix}, \tag{2.1}$$

where $a\varepsilon = b$ if $a \neq 0$. We then have three parameters, a, b and z, or a, ε and z. In this section, we shall partition the parameter spaces $\mathcal{P}_3 = \{(z, a, b) :, a, b, z \in \mathcal{R}\}$ or $= \{(z, a, \varepsilon) : a, \varepsilon, z \in \mathcal{R}\}$ into finitely many regions such that in each region, (1.1) has the same mosaic patterns.

From now on, we shall assume (2.1) holds. When $a \neq 0$ and x is a solution, then for any $(i,j) \in \mathcal{Z}^2$, $(x_{i,j}, y_{i,j})$ will satisfy

$$y = f(x) \tag{2.2}$$

and

$$y = \frac{1}{a}\{x - (z + 2kb)\}, \tag{2.3a}$$

or

$$y = \tfrac{1}{a}\{x - (z + 2ka\varepsilon)\}, \tag{2.3b}$$

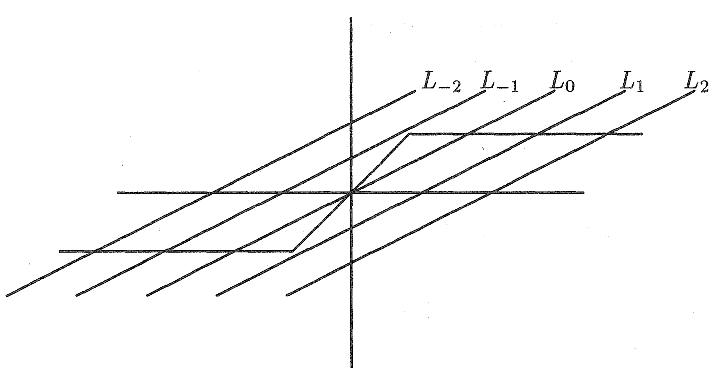
for $k \in \{-2, -1, 0, 1, 2\}$, i.e., $(x_{i,j}, f(x_{i,j}))$ lies on one of the five straight lines $L_{k,\varepsilon}$ defined in (2.3), where z, a and b are fixed. For $a = 0$, (3.3) reduces to

$$x - z - 2kb = 0. \tag{3.3c}$$

Note that when $k = 2$, this corresponds to an unknown cell $C_{i,j}$ being surrounded by 4 positively saturated cells. Similar interpretations can be applied to $k = 1, 0, -1, -2$.

To pursue this idea for partitioning $\mathcal{P}_3$ in more detail, we first need the following notation.

Definition 2.1. For any two integers $k < \ell$, denote $I[k, \ell] = \{k, k+1, \cdots, \ell\}$, the set of integers that are no greater than ℓ and no smaller than k. For $m, n \in I[0, 5]$, denote $[m, n]$ the (open) subset of $\mathcal{P}_3$ such that the intersection of (3.2) and (3.3) consists of m positively saturated states; e.g., $(x > 1)$ and there are n negatively saturated states; e.g., $(x < -1)$. Furthermore, for any fixed b or ε, we may also use $[m, n]$, or $[m, n]_b$ or $[m, n]_\varepsilon$ if necessary, to describe such an open subset in $\mathcal{P}_2 = \{(z, a) : z, a \in \mathcal{R}\}$. See, Fig 1.



$a\varepsilon > 0, \ m = 3 = n.$

Figure 1

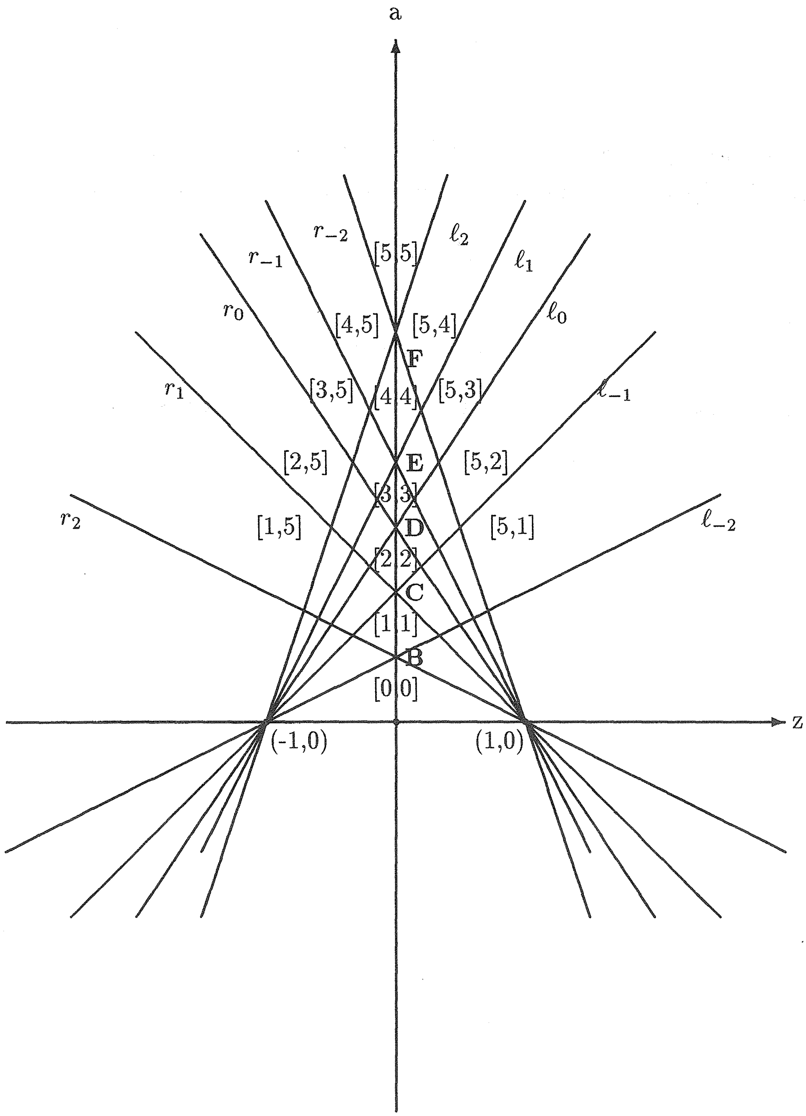
It is much easier to partition $\mathcal{P}_2$ into $[m,n]_\varepsilon$ by fixing and then varying $\varepsilon \in \mathcal{R}$. Indeed, for each ε and $k \in I[-2,2]$, let $r_{k,\varepsilon}$ and $\ell_{k,\varepsilon}$ be straight lines whose equations are

$$r_{k,\varepsilon} : z + (1+2k\varepsilon)a = 1, \tag{2.3}$$

and

$$\ell_{k,\varepsilon} : -z + (1-2k\varepsilon)a = 1. \tag{2.4}$$

We draw the result as in Fig 2, for $0 < |\varepsilon| < \frac{1}{4}$, the other cases can be treated analogously.



$$B = \frac{1}{1+4|\varepsilon|}, \quad C = \frac{1}{1+2|\varepsilon|}, \quad D = 1, \quad E = \frac{1}{1-2|\varepsilon|}, \quad F = \frac{1}{1-4|\varepsilon|}$$

$$0 < |\varepsilon| < \frac{1}{4}.$$

Fig 2.

3. Mosaic solutions

For each $[m, n]$, we begin with the study of feasible local patterns. Using these feasible patterns, we can form a set of building blocks that can be glued together according certain rules (compatibility conditions) to construct all mosaic patterns.

The set of nearest neighbors to the point (i, j) is defined by

$$N^+(i,j) = \{(i+k, j+\ell) \in \mathbf{Z}^2 : |k| + |\ell| = 1\}.$$

We have the basic result for $[m, n]_\varepsilon$ as follows.

Lemma 3.1. (Existence or Feasibility Lemma for $[m, n]_\varepsilon$).
Given parameters z, a, and ε in $[m, n]_\varepsilon$, and that $a\varepsilon > 0$, $x = (x_{ij})$ is a feasible (or stable) solution if and only if any positively (resp., negatively) saturated cells must be coupled to at least $5 - m$ positively (resp., $5 - n$ negatively) saturated cells. On the other hand, if $a\varepsilon < 0$, then any positively (resp., negatively) saturated cell must be coupled to at least $5 - m$ negatively (resp., $5 - n$ positively) saturated cells.

Note that the constraints given in Lemma 3.1 are basic, and also that only rule must be obeyed in obtaining a global pattern. We next introduce the following feasibility conditions for local patterns for which we need the following notation.

Definition 3.2. Given any (proper) subset $T \subseteq \mathcal{Z}^2, x(\equiv x_T)$ is called a local solution if x_T is a restriction of some mosaic solution x of (2.1) on T. Similarly, $y(\equiv y_T) : T \to \{-1, 1\}$ is called a local pattern if it is an output of some (local) solution x of (2.1) on T. When $T = \mathcal{Z}^2$, y is called a global pattern. A set $T \subseteq \mathcal{Z}^2$ is called basic with respect to the template A if $T = T_{i,j} \equiv \{(i,j)\} \cup N^+(i,j)$ for some $(i,j) \in \mathcal{Z}^2$. A basic pattern (BP) y is a feasible pattern defined on some basic set.

Denote by $\mathcal{F}([m, n])$, the set of all feasible basic patterns that have parameters in $[m, n]$. An easy consequence of Lemma 3.1 is the following assertion.

Proposition 3.3. For any $[m, n], \mathcal{F}([m, n])$ is unique and finite.

We now give a partial list of possible $\mathcal{F}([m, n])$.

Propositions 3.4. Given a set of (local or global) patterns $Y = \{y_\alpha\}$, we denote by $R(Y)$ the set of all patterns that are rotated by multiples of 90° from original patterns in Y. Suppose $\bullet$ is either $+$ or $-$. Then

$$\text{(i)} \qquad \mathcal{F}([5,5]) = \left\{ \begin{array}{ccc} & \bullet & \\ \bullet & + & \bullet \\ & \bullet & \end{array}, \begin{array}{ccc} & \bullet & \\ \bullet & - & \bullet \\ & \bullet & \end{array} \right\},$$

$$\text{(ii)} \qquad \mathcal{F}([4,4]) = R\left\{ \begin{array}{ccc} & \bullet & \\ \bullet & + & + \\ & \bullet & \end{array}, \begin{array}{ccc} & \bullet & \\ \bullet & - & - \\ & \bullet & \end{array} \right\},$$

$$\text{(iii)} \qquad \mathcal{F}([3,3]) = R\left\{ \begin{array}{ccc} & + & \\ + & + & \bullet \\ & \bullet & \end{array}, \begin{array}{ccc} & + & \\ \bullet & + & \bullet \\ & + & \end{array}, \begin{array}{ccc} & - & \\ - & - & \bullet \\ & \bullet & \end{array}, \begin{array}{ccc} & - & \\ \bullet & - & \bullet \\ & - & \end{array} \right\},$$

$$\text{(iv)} \qquad \mathcal{F}([3,2]) = R\left\{ \begin{array}{ccc} & + & \\ + & + & \bullet \\ & \bullet & \end{array}, \begin{array}{ccc} & + & \\ \bullet & + & \bullet \\ & + & \end{array}, \begin{array}{ccc} & - & \\ - & - & - \\ & \bullet & \end{array} \right\},$$

$$\text{(v)} \qquad \mathcal{F}([2,2]) = R\left\{ \begin{array}{ccc} & + & \\ + & + & + \\ & \bullet & \end{array}, \begin{array}{ccc} & - & \\ - & - & - \\ & \bullet & \end{array} \right\},$$

$$\text{(vi)} \qquad \mathcal{F}([1,1]) = \left\{ \begin{array}{ccc} & + & \\ + & + & + \\ & + & \end{array}, \begin{array}{ccc} & - & \\ - & - & - \\ & - & \end{array} \right\},$$

$$\text{(vii)} \qquad \mathcal{F}([1,0]) = \left\{ \begin{array}{ccc} & + & \\ + & + & + \\ & + & \end{array} \right\},$$

$$\text{(viii)} \qquad \mathcal{F}([0,0]) = \phi.$$

We can glue two BP's together if they follow the rule given in Lemma 3.1. However, to construct all global mosaic patterns for each $[m,n]$, we need to find a more efficient way to glue appropriate feasible patterns tegether than using BP alone. To this end, we must introduce the concept of building blocks and compatibility conditions for patching them together.

Definition 3.5. Let $\mathcal{P} \subset \mathcal{P}_3$ be a set of parameters in $\mathcal{P}_3$. $\mathcal{B} = \mathcal{B}(\mathcal{P})$ a (finite or infinite) set of feasible local patterns, is called a set of building blocks provided that every global mosaic pattern in $\mathcal{M}(\mathcal{P})$ can be generated by patching these building blocks together with respect to some compatibility condition $\mathcal{C}(\mathcal{P})$.

If $\mathcal{P} = [m,n]$, we write $\mathcal{B}(\mathcal{P})$ as $\mathcal{B}([m,n])$, and $\mathcal{C}(\mathcal{P})$ as $\mathcal{C}([m,n])$. Note that for a given $\mathcal{P}$, $\{\mathcal{B}(\mathcal{P}), \mathcal{C}(\mathcal{P})\}$ is not necessarily unique if it does exist. However, we would like to have $\{\mathcal{B}(\mathcal{P}), \mathcal{C}(\mathcal{P})\}$ be such that as few elements as possible are in $\mathcal{B}(\mathcal{P})$, and rule $\mathcal{C}(\mathcal{P})$ is as simple as possible, since they are related to the transition matrices used to compute spatial entropy of $\mathcal{M}(\mathcal{P})$. Sometimes, a natural and obvious way can be used to find $\{\mathcal{B}(\mathcal{P}), \mathcal{C}(\mathcal{P})\}$ for certain $\mathcal{P}$. In general, finding an efficient and effective $\{\mathcal{B}(\mathcal{P}), \mathcal{C}(\mathcal{P})\}$ in order to compute the entropy $h(\mathcal{M}(\mathcal{P}))$ is a form of art; for which we need the following notion.

Definition 3.6. Let $y_j : T_j \to \{-1, 1\}, j = 1, 2$, be two feasible local patterns with $T_1 \cap T_2 \neq \phi$. y_1 and y_2 then are called compatible if

$$y_1 = y_2 \quad \text{on} \quad T_1 \cap T_2.$$

We say two feasible local patterns $y_j : T_j \to \{-1, 1\}, j = 1, 2$, is adjacent to another if $T_1 \cap T_2 = \phi$ and at least one cell from each set $T_j, j = 1, 2$, is adjacent to another.

We give the following simple compatibility rules to generate larger local patterns.

$\mathcal{C}_0$: Put any two feasible local patterns y_1 and y_2 in $\mathcal{B}(\mathcal{P})$ that are adjacent to each other together.

$\mathcal{C}_1$: Glue together any two feasible local patterns y_1 and y_2 in $\mathcal{B}(\mathcal{P})$ that are compatible.

Note that the feasibility $y_1 \cup y_2$ of both cases has to be verified. In practice, it is easy to check this by using BP in $\mathcal{F}([m,n])$. For simplicity, we only state our result for [5, 5] and [4, 4].

Theorem 3.7.

(I) $\mathcal{B}([5,5]) = \{+, -\}$ and $\mathcal{C}([5,5]) = \mathcal{C}_0$.

(II) (i) If $a\varepsilon > 0, \mathcal{B}([4,4]) = R\{++, --\}$, and $\mathcal{C}([4,4]) = \mathcal{C}_0 \cup \mathcal{C}_1$.

(ii) If $a\varepsilon < 0, \mathcal{B}([4,4]) = R\{+-\}$, and $\mathcal{C}([4,4]) = \mathcal{C}_0 \cup \mathcal{C}_1$.

As for the result for spatial complexity, we have the following results.

Theorem 3.8. Let $m, n \in I[0,5]$, and let

$$\alpha = max\{m,n\} \quad \text{and} \quad \beta = min\{m,n\}. \tag{3.1}$$

(1.1) then exhibits spatial chaos if and only if $\alpha \geq 3$ and $\beta \geq 2$.

Proof: It is clear $\mathcal{M}([m,n])$ is monotonous with respect to m and n, e.g., if $m_1 \leq m_2$ and $n_1 \leq n_2$, then

$$\mathcal{M}([m_1, n_1]) \subseteq \mathcal{M}([m_2, n_2]).$$

To prove the theorem, it suffices to show only that

$$h(\mathcal{M}([2,2])) = 0, \tag{3.2}$$

$$\text{and} \qquad h(\mathcal{M}([3,2])) > 0. \tag{3.3}$$

We first prove (3.2). Let $N = (N_1, 2)$ and $N_1 \geq 2$, we then have

$$\Gamma_N(\mathcal{M}([2,2])) \leq 4,$$

here $\Gamma_N(u)$ is the number of distinct patterns obervable among the element of u restriction on the rectangle N. Hence (3.2) holds.

To prove (3.3), we may assume $a\varepsilon > 0$, the case in which $a\varepsilon < 0$ can be treated analogously. Consider a rectangle of size $4n_1 \times 4n_2$ in $\mathcal{Z}^2$. So, there are $n_1 \cdot n_2$ many squares of size 4×4.

Consider the following choices of patterns for a 4×4 square:

$$\begin{array}{cccc} - & - & - & - \\ - & - & - & - \\ + & + & - & - \\ + & + & - & - \end{array} \qquad \begin{array}{cccc} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{array}. \tag{3.4}$$

They are feasible and compatible with each other in $[3,2]$. Therefore, they can be glued together at random. Hence, for $N = (4n_1, 4n_2)$, we have

$$\Gamma_N(\mathcal{M}([3,2])) \geq 2^{n_1 n_2}. \tag{3.5}$$

From (3.5), it is not difficult to prove that

$$h(\mathcal{M}([3,2])) \geq \frac{\log 2}{16}. \tag{3.6}$$

The proof of the theorem is thus complete. ∎

Furthermore, we can obtain some lower bounds for $h(\mathcal{M}([m,n]))$. When (3.1) holds, some lower bounds for $h(\mathcal{M}([m,n]))$ can be obtained by the following.

Theorem 3.9.

$$h(\mathcal{M}([m,n])) \geq \begin{cases} log\ 2 & \text{if} \quad \beta = 5, \\ \frac{log\ 10}{4} & \text{if} \quad \beta = 4, \\ \frac{log\ 4}{4} & \text{if} \quad \beta = 3, \\ \frac{log\ 4}{9} & \text{if} \quad \beta = 2, \alpha = 5, \\ \frac{log\ 4}{12} & \text{if} \quad \beta = 2, \alpha = 4, \\ \frac{log\ 2}{16} & \text{if} \quad \beta = 2, \alpha = 3. \end{cases} \tag{3.7}$$

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