THE L^p BOUNDEDNESS OF RIESZ TRANSFORMS ASSOCIATED WITH DIVERGENCE FORM OPERATORS

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ABSTRACT. Let A be a divergence form elliptic operator associated with a quadratic form on Ω where Ω is the Euclidean space $\mathbb{R}^n$ or a domain of $\mathbb{R}^n$. Assume that A generates an analytic semigroup e^{-tA} on $L^2(\Omega)$ which has heat kernel bounds of Poisson type, and that the generalised Riesz transform $\nabla A^{-1/2}$ is bounded on $L^2(\Omega)$. We then prove that $\nabla A^{-1/2}$ is of weak type $(1,1)$, hence bounded on $L^p(\Omega)$ for $1 < p \leq 2$. No specific assumptions are made concerning the Hölder continuity of the coefficients or the smoothness of the boundary of Ω.

1. Introduction.

Let Ω be the Euclidean space $\mathbb{R}^n$ or a domain of $\mathbb{R}^n$. In the latter case, no smoothness condition is assumed on the boundary of Ω unless it is implied by other assumptions. Let Q be the sesquilinear form on the product space $\mathcal{V} \times \mathcal{V}$, where $C_0^\infty(\Omega) \subset \mathcal{V} \subset W^{1,2}(\Omega)$, given by

$$(1) \qquad Q(f,g) = \int_\Omega \sum_{i,j} a_{ij}(x) \frac{\partial f}{\partial x_j} \frac{\partial \overline{g}}{\partial x_i} dx$$

for $f, g \in \mathcal{V}$, and a_{ij} are bounded, measurable, complex-valued coefficients which satisfy

$$(2) \qquad |\Im m \sum_{i,j} a_{ij}(x)\zeta_j \overline{\zeta}_i| \leq C \, \Re e \sum_{i,j} a_{ij}(x)\zeta_j \overline{\zeta}_i$$

for some constant C, for all $\zeta \in \mathbb{C}^n$ and almost all $x \in \Omega$. We also assume the uniform ellipticity conditions

$$(3) \qquad \delta|\zeta|^2 \leq \Re e \sum_{i,j} a_{ij}(x)\zeta_j \overline{\zeta}_i \leq \kappa|\zeta|^2$$

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where δ and κ are positive constants uniformly for $x \in \Omega$ and $\zeta \in \mathbb{C}^n$.

Let A be the divergence form operator associated with the form Q in the sense that A is the operator in $L^2(\Omega)$ with largest domain $\mathcal{D}(A)$ which satisfies

(4) $$< Au, g > = Q(u, g)$$

for all $u \in \mathcal{D}(A)$ and all test functions $g \in \mathcal{V}$. Different choices of the space $\mathcal{V}$ give the operator A different corresponding boundary value conditions when Ω is a domain in $\mathbb{R}^n$. For example, when $\mathcal{V}$ is $W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)$, it corresponds to Dirichlet boundary conditions and Neumann boundary conditions, respectively.

The operator A generates a bounded holomorphic semigroup e^{-zA}, $|\arg z| < \mu$ for some $\mu < \frac{\pi}{2}$. See [K], Chapter 9.

In this paper, we consider the generalised Riesz transform $T = \nabla A^{-1/2}$ associated with the divergence form operator A, defined by

$$Tu = \frac{1}{2\sqrt{\pi}} \int_0^\infty \nabla e^{-sA} u \frac{ds}{\sqrt{s}}.$$

Note that if A is the Laplacian $-\Delta$ on $\mathbb{R}^n$, then T is the classical Riesz transform. A natural question is the boundedness of T on L^p spaces. In [AT], it is proved that when $\Omega = \mathbb{R}^n$, the operator T is bounded from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, hence by interpolation, is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq 2$, under the following assumptions:

(α) The analytic semigroup e^{-tA} generated by A has kernels which possess Gaussian upper bounds and Hölder continuity bounds in their space variables; and

(β) $T = \nabla A^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$.

For more details, see [AT] Chapter 4, Theorem 1.

The aim of this paper is to prove the L^p boundedness of T without the assumption of Hölder continuity in the space variables of the kernels of the semigroup e^{-tA}. An important consequence of removing the assumption of Hölder continuity is that we can obtain positive results when Ω is a domain of $\mathbb{R}^n$ without any assumptions on the smoothness of its boundary. Thus the usual Calderón-Zygmund operator theory is not directly applicable for two reasons: the kernels may not be Hölder continuous, and the domain may not be of homogeneous type.

We overcome these problems by using recent results on boundedness of singular integrals in [DMc]. We also use the idea in [CD1] of proving a weighted L^2 estimate. This paper [CD1] of Coulhon and Duong carried out a similar program for the Laplace–Beltrami operator on a Riemannian manifold. It also treated boundary value problems for the Laplacian on domains in $\mathbb{R}^n$, generalising earlier results in [JK].

2. Singular integrals.

Let Ω be the space $\mathbb{R}^n$ or a connected open subset of $\mathbb{R}^n$, equipped with Euclidean distance and Lebesgue measure. We assume no smoothness assumption on the boundary of Ω unless it is implied by other assumptions. For such an Ω, there exists a subset X such that $\Omega \subseteq X \subseteq \mathbb{R}^n$ and the space X equipped with Euclidean distance and Lebesgue measure is a space of homogeneous type. This means that X has the doubling volume property

$$|B^X(x; 2r)| \leq \alpha\, |B^X(x; r)|$$

where α is a constant uniformly for all $x \in X$, $r > 0$, and $|B^X(x; r)|$ denotes the Lebesgue measure of the ball in X with centre x, radius r.

The doubling volume property implies the following strong homogeneity property

$$|B^X(x; \lambda r)| \leq \alpha_1 \lambda^n |B^X(x; r)|$$

for some α_1 uniformly for all $\lambda \geq 1$, with n the dimension of the space. It also implies the following result.

Lemma 1. *Given* $\gamma > \frac{n}{2}$, *then*

$$\int_\Omega \left(1 + \frac{|x - y|^2}{s}\right)^{-\gamma} dx \leq c\, |B^X(y; \sqrt{s})|, \quad y \in \Omega.$$

Proof.

$$\int_\Omega \left(1 + \frac{|x - y|^2}{s}\right)^{-\gamma} dx \leq \int_{B^X(y,\sqrt{s})} \left(1 + \frac{|x - y|^2}{s}\right)^{-\gamma} dx +$$

$$\sum_{k=0}^{\infty} \int_{2^k \sqrt{s} \leq |x-y| \leq 2^{k+1}\sqrt{s}} \left(1 + \frac{|x - y|^2}{s}\right)^{-\gamma} dx$$

$$\leq c\, |B^X(y; \sqrt{s})| + \sum |B^X(y; 2^{k+1}\sqrt{s})|\, (1 + 2^{2k})^{-\gamma}$$

$$\leq c\, |B^X(y; \sqrt{s})|$$

by the above homogeneity property. $\square$

One obvious choice of such an X is R^n itself, though it is preferable to choose X smaller if possible. See Conditions (6) and (7) in Theorem 1, Conditions (9) and (10) in Section 3, and the application in Section 4.

Let T be a bounded linear operator on $L^2(\Omega)$ with an associated kernel $k(x,y)$ in the sense that

$$(5) \qquad (Tf)(x) = \int_\Omega k(x,y)f(y)dy$$

where $k(x,y)$ is a measurable function, and the above formula holds for each continuous function f with compact support, and for almost all x not in the support of f.

The following theorem is essentially Theorem 2, [DMc].

Theorem 1. *Let T be a bounded linear operator from $L^2(\Omega)$ to $L^2(\Omega)$. Assume there exists a class of operators A_t, $t > 0$, defined on $L^2(\Omega)$ which is represented by kernels $a_t(x,y)$ in the sense that*

(a) $A_t u(x) = \int_\Omega a_t(x,y)u(y)dy$ for any function $u \in L^2(\Omega) \bigcap L^1(\Omega)$, and the kernels $a_t(x,y)$ satisfy the following conditions:

$$(6) \qquad |a_t(x,y)| \le h_t(x,y)$$

for all $x,y \in \Omega$, where $h_t(x,y)$ is defined on $X \times X$ by

$$(7) \qquad h_t(x,y) = (|B^X(y;t^{1/m})|)^{-1} s(|x-y|^m t^{-1}),$$

for some $m > 0$, and s is a positive, bounded, decreasing function satisfying

$$\lim_{r \to \infty} r^{n+\epsilon} s(r^m) = 0$$

for some $\epsilon > 0$, and n the dimension of the space.

(b) the operators $T - TA_t$ have associated kernels $K_t(x,y)$ in the sense of (5), and there exist constants C and $c > 0$ so that

$$(8) \qquad \int_{\{x \in \Omega : |x-y| \ge ct^{1/m}\}} |K_t(x,y)|dx \le C, \quad y \in \Omega.$$

Then the operator T is of weak type $(1,1)$. Hence T can be extended from $L^2(\Omega) \bigcap L^p(\Omega)$ to a bounded operator on $L^p(\Omega)$ for all $1 < p \le 2$.

NOTES:

(α) The class of operators A_t plays the role of approximations to the identity. Note that we have no smoothness assumption on the space variables of a_t.

(β) Typical examples of the bound $h_t(x, y)$ on $R^n \times R^n$ are the Gaussian bound

$$ct^{-n/2} \exp\{-\alpha|x - y|^2/t\}$$

and the Poisson bound

$$\frac{ct}{(t^2 + |x - y|^2)^{(n+1)/2}}.$$

(γ) The key estimate needed in Theorem 2 is Condition (8). In a space of homogeneous type, this condition is strictly weaker than the usual Hörmander condition for weak type $(1, 1)$ estimates:

$$\int_{|x-y|\geq\delta|y_1-y|} |k(x, y) - k(x, y_1)|dx \leq C$$

for all $y, y_1 \in X$. See Proposition 1, [DMc].

(δ) The uniform upper bound in (7) is in terms of volumes of the balls in the space of homogeneous type X, not the domain Ω. This plays an essential role in overcoming the problem that Ω does not satisfy the doubling volume property.

(ϵ) Theorem 1 builds on an earlier result about L^p boundedness of functional calculi [DR].

3. Boundedness of generalised Riesz transforms.

Let A be the divergence form operator associated with a sesquilinear form Q as in (1) to (4). We assume that

(i) The analytic semigroup e^{-tA} generated by A has kernels $p_t(x, y)$ with Poisson type upper bounds, that is

(9) $$|p_t(x, y)| \leq c\, h_t(x, y)$$

for all $t > 0$, and all $x, y \in \Omega$, where $h_t(x, y)$ is defined on $X \times X$ by

(10) $$h_t(x, y) = \frac{1}{|B^X(y; \sqrt{t})|(1 + \frac{|x-y|^2}{t})^\beta},$$

where c is a constant, $\beta > \frac{n}{2}$, and B^X is a ball in a space of homogeneous type X such that $\Omega \subseteq X \subseteq \mathbb{R}^n$. Such functions h_t satisfy Condition (7) with $m = 2$.

(ii) The operator $T = \nabla A^{-1/2}$ defined by

$$Tu = \frac{1}{2\sqrt{\pi}} \int_0^\infty \nabla e^{-sA} u \frac{ds}{\sqrt{s}}$$

is bounded on $L^2(\Omega)$. This holds if and only if the domain $\mathcal{D}(A^{1/2}) \subseteq \mathcal{V}$ with $\|\nabla u\|_2 \leq c \|A^{1/2}u\|_2$.

(iii) The space $\mathcal{V}$, the domain of the sesquilinear form, is invariant under multiplication by bounded functions with bounded, continuous first derivatives. This condition is satisfied by Dirichlet, Neumann and mixed boundary conditions.

This last condition could no doubt be weakened by appropriately adapting the weights w_t used in the proof.

Our first observation is that condition (i) implies that the semigroup e^{-tA} has estimates on the time derivatives of its kernels. More specifically, we have the following result.

Lemma 2. *Let T_t be a uniformly bounded analytic semigroup on $L^2(\Omega)$ and assume that T_t, $t > 0$, has a kernel $p_t(x,y)$ satisfying*

$$|p_t(x,y)| \leq \frac{c}{|B^X(y; \sqrt{t})|(1 + \frac{|x-y|^2}{t})^\beta}.$$

and let $\frac{n}{2} < \gamma < \beta$. Then the time derivatives $\frac{d^k}{dt^k}T_t$ have kernels $\frac{d^k}{dt^k}p_t$ which satisfy

$$\left|\frac{d^k}{dt^k}p_t(x,y)\right| \leq \frac{c_k}{t^k |B^X(y; \sqrt{t})|(1 + \frac{|x-y|^2}{t})^\gamma}, \quad x,y \in \Omega.$$

Proof. This proof has three steps. First, we prove uniform bounds on complex heat kernels, then use the Poisson formula on the half plane to obtain Poisson bounds on complex heat kernels. The desired estimates on $\frac{d^k}{dt^k}p_t(x,y)$ follow from a standard Cauchy formula. See Section 2.1 [CD2], [DR] and their references for details. □

It follows from the boundedness of the operators $\frac{d^k}{dt^k}e^{-tA} = A^k e^{-tA}$ on the L^2 space that the heat kernels $p_t(.,y)$ and their time derivatives $\frac{d}{dt}p_t(.,y)$ belong to the domain of the operator A. In particular, $p_t(.,y) \in \mathcal{V}$ and so $\nabla p_t(.,y) \in L^2(\Omega)$.

We now introduce a family of smooth, bounded, radially increasing weight functions $w_t^R(x, y)$, $t > 0$, $R \geq 2$, as follows:

$$w_t^R(x, y) = \begin{cases} (1 + \frac{|x-y|^2}{t})^{\frac{n}{2}+\epsilon}, & |x - y| \leq R\sqrt{t}, \\ (1 + (2R)^2)^{\frac{n}{2}+\epsilon}, & |x - y| \geq 2R\sqrt{t} \\ \text{smooth, increasing when} & R\sqrt{t} \leq |x - y| \leq 2R\sqrt{t} \end{cases}$$

where $0 < \epsilon < \beta - \frac{n}{2}$. The weight functions can be chosen so that their space derivatives satisfy the condition

(11)
$$\left| \frac{\partial w_t^R(x, y)}{\partial x_j} \right| \leq \frac{c}{\sqrt{t}} w_t^R(x, y).$$

We also use the weight functions $w_t(x, y) = (1 + \frac{|x-y|^2}{t})^{\frac{n}{2}+\epsilon}$. It is straightforward that the estimate (11) is satisfied with $w_t(x, y)$ in place of $w_t^R(x, y)$. We comment that, using Lemma 1,

$$\int_{|x-y| \geq \sqrt{t}} \frac{1}{w_s(x, y)} \, dx \leq \sup_{|x-y| \geq \sqrt{t}} \frac{1}{(1 + \frac{|x-y|^2}{s})^{\epsilon/2}} \int_{\Omega} \frac{1}{(1 + \frac{|x-y|^2}{s})^{n/2+\epsilon/2}} \, dx$$

(12)
$$\leq \frac{c \, |B^X(y, \sqrt{s})|}{(1 + \frac{t}{s})^{\epsilon/2}}.$$

We show a key estimate on the space derivative of $p_t(x, y)$ in the next lemma.

Lemma 3. *Let*

$$J_t(y) = \int_{\Omega} |\nabla_x p_t(x, y)|^2 w_t(x, y) \, dx, \quad y \in \Omega.$$

Then $J_t(y)$ is finite and satisfies

$$J_t(y) \leq \frac{c}{t \, |B^X(y, \sqrt{t})|}.$$

Proof. We use an idea in [CD1]. First consider

$$J_t^R(y) = \int_{\Omega} |\nabla_x p_t(x, y)|^2 w_t^R(x, y) \, dx.$$

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Then, by the ellipticity assumptions,

$$J_t^R(y) \approx \left| \int_\Omega \sum_{i,j} a_{ij}(x) \frac{\partial p_t(x,y)}{\partial x_j} \frac{\partial \bar{p}_t(x,y)}{\partial x_i} w_t^R(x,y) \ dx \right|$$

$$= \left| Q(p_t(\,.\,,y)\,,\, p_t(\,.\,,y)w_t^R(\,.\,,y)) - \int_\Omega \sum_{i,j} a_{ij}(x) \frac{\partial p_t(x,y)}{\partial x_j} \bar{p}_t(x,y) \frac{\partial w_t^R(x,y)}{\partial x_i} \ dx \right| .$$

We noted above that $p_t(\,.\,,y) \in \mathcal{V}$, and so by assumption (iii), $p_t(\,.\,,y)w_t^R(\,.\,,y) \in \mathcal{V}$ also. (This is one reason for introducing w_t^R rather than treating w_t directly.) Also $\nabla p_t(\,.\,,y) \in L^2(\Omega)$. Therefore, for each $y \in \Omega$, the quantity $J_t^R(y) < \infty$.

Since $p_t(\,.\,,y) \in \mathcal{D}(A)$,

$$\left| Q(p_t(\,.\,,y)\,,\, p_t(\,.\,,y)w_t^R(\,.\,,y)) \right| = \left| < A p_t(\,.\,,y)\,,\, p_t(\,.\,,y)w_t^R(\,.\,,y) > \right|$$

$$= \left| \int_\Omega \frac{d}{dt} p_t(x,y) \ \bar{p}_t(x,y) w_t^R(x,y) \ dx \right|$$

$$\leq \frac{c}{t \, |B^X(y,\sqrt{t})|} .$$

This inequality is a consequence of the upper bounds on the heat kernels and their time derivatives, and of Lemma 1.

Further, using (11) and Lemma 1, we have

$$\left| \int_\Omega \sum_{i,j} a_{ij}(x) \frac{\partial p_t(x,y)}{\partial x_j} \bar{p}_t(x,y) \frac{\partial w_t^R(x,y)}{\partial x_i} \ dx \right|$$

$$\leq \frac{c}{\sqrt{t}} \left(\int_\Omega |\bar{p}_t(x,y)|^2 w_t^R(x,y) \ dx \right)^{1/2} \left(\int_\Omega |\nabla_x p_t(x,y)|^2 w_t^R(x,y) \ dx \right)^{1/2}$$

$$\leq \frac{c}{\sqrt{t} \, |B^X(y,\sqrt{t})|^{1/2}} \left(\int_\Omega |\nabla_x p_t(x,y)|^2 w_t^R(x,y) \ dx \right)^{1/2} .$$

Hence

$$J_t^R(y) \leq \frac{c}{t \, |B^X(y,\sqrt{t})|} + \frac{c}{\sqrt{t} \, |B^X(y,\sqrt{t})|^{1/2}} \sqrt{J_t^R(y)} .$$

Therefore, using the fact already proved that $J_t^R(y) < \infty$, it follows that

(13)
$$J_t^R(y) \leq \frac{c}{t \, |B^X(y,\sqrt{t})|} .$$

We conclude that

$$J_t(y) = \int_\Omega |\nabla_x p_t(x,y)|^2 w_t(x,y)\, dx$$

$$= \sup_R \int_{|x-y|\leq R} |\nabla_x p_t(x,y)|^2 w_t(x,y)\, dx = \sup_R \int_\Omega |\nabla_x p_t(x,y)|^2 w_t^R(x,y)\, dx$$

$$\leq \frac{c}{t\,|B^X(y,\sqrt{t})|} \qquad \text{(using (13)).} \quad \square$$

The main result of this paper is the following theorem.

Theorem 2. *Under the above assumptions (i), (ii), and (iii), the operator $T = \nabla A^{-1/2}$ is of weak type $(1,1)$. Hence T can be extended to a bounded operator on $L_p(\Omega)$ for $1 < p \leq 2$.*

Proof. It is sufficient to prove that the conditions in Theorem 1 are satisfied with $m = 2$ and an appropriate choice of operators A_t.

Choose $A_t = e^{-tA}$. Conditions (6) and (7) are satisfied because of assumption (i).

Let $K_t(x,y)$ be the kernels of the operators $T(I - e^{-tA})$. We just need to check Condition (8), i.e.

$$(14) \qquad \int_{|x-y|\geq\sqrt{t}} |K_t(x,y)|\, dx \leq C\,, \quad y \in \Omega.$$

Let us compute the kernel $K_t(x,y)$. We have

$$\nabla A^{-1/2}(I - e^{-tA}) = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \nabla e^{-sA} \frac{ds}{\sqrt{s}} - \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \nabla e^{-(s+t)A} \frac{ds}{\sqrt{s}}$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \left(\frac{1}{\sqrt{s}} - \frac{\chi_{\{s>t\}}}{\sqrt{s-t}} \right) \nabla e^{-sA}\, ds.$$

Therefore

$$(15) \qquad K_t(x,y) = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \left(\frac{1}{\sqrt{s}} - \frac{\chi_{\{s>t\}}}{\sqrt{s-t}} \right) \nabla_x p_s(x,y)\, ds\,.$$

Thus the key to proving (14) is a good estimate of

$$(16) \qquad \int_{|x-y|\geq\sqrt{t}} |\nabla_x p_s(x,y)|\, dx.$$

Using the weight functions $w_s(x,y)$ introduced previously, we have

$$\int_{\{x\in\Omega:|x-y|\geq\sqrt{t}\}} |\nabla_x p_s(x,y)|\, dx$$

$$\leq \left(\int_{|x-y|\geq\sqrt{t}} \frac{1}{w_s(x,y)}\, dx\right)^{1/2} \left(\int_\Omega |\nabla_x p_s(x,y)|^2 w_s(x,y)\, dx\right)^{1/2}$$

$$\leq \left(\frac{c\,|B^X(y,\sqrt{s})|}{(1+\frac{t}{s})^{\epsilon/2}}\right)^{\frac{1}{2}} \left(\frac{c}{s\,|B^X(y,\sqrt{s})|}\right)^{\frac{1}{2}} \leq \frac{c}{\sqrt{s}}\frac{1}{(1+\frac{t}{s})^{\epsilon/2}}$$

by (12) and Lemma 3.

Combining this with (15), elementary integration shows that (14) is satisfied. This completes the proof. $\square$

NOTES:

(a) The assumption (i) of Theorem 2 on heat kernel bounds is satisfied by large classes of divergence form operators on $\mathbb{R}^n$ or a domain of $\mathbb{R}^n$. In the case of Dirichlet boundary conditions, we can have Gaussian heat kernel bounds without any conditions on smoothness of the boundary of Ω. In the case of Neumann boundary conditions, we need the domain Ω to possess the extension property to ensure heat kernel bounds even with the Laplacian. For example, see [Da], [AE] for divergence form operators with real coefficients, [AMᶜT] and [A] for certain operators with complex coefficients.

(b) The usual methods imply that T is bounded on $L^p(\Omega)$ when $1 < p < \beta$ for some $\beta > 2$. On the other hand, there are counter-examples due to Kenig which show that for any $\alpha > 2$, there exist divergence form operators on $\mathbb{R}^n$ which satisfy the hypotheses of Theorem 2 but whose associated Riesz transforms are not bounded on L^p for $p > \alpha$. See for example [AT] Chapter 4.

(c) If the assumption (ii) is replaced by the statement that T is bounded on $L^q(\Omega)$ for some $q > 1$, then Theorem 2 remains valid for $1 < p \leq q$.

4. An Application.

Let us conclude with one specific consequence of Theorem 2. Clearly its applicability goes well beyond this.

Theorem 3. *Let A be the divergence form operator with Neumann boundary conditions associated with a sesquilinear form Q as in (1) to (4), where $\mathcal{V} = W^{1,2}(\Omega)$. Suppose that the coefficients are real and satisfy $a_{ij} = a_{ji}$, and that Ω is a bounded*

domain which satisfies the extension property. Then the Riesz transform $T = \nabla A^{-1/2}$ is a bounded operator on $L^p(\Omega)$, $1 < p \le 2$.

Proof. Let X be a ball in $\mathbb{R}^n$ which contains Ω, so that

$$\frac{1}{|B^X(y; \sqrt{t})|} \approx \max\{1, t^{-n/2}\} \, .$$

The assumption (i) is proved under these conditions in [Da], Theorem 3.2.9, while assumption (ii) follows from the fact that Q is a positive Hermitian form ([K], Theorem VI.2.23). Therefore Theorem 2 can be applied to give the result. $\square$

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