SOME LUSIN PROPERTIES OF FUNCTIONS

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This note will complement our recent works in [9], [10], and [11] on Lusin properties of functions. Let D be a Lebesgue measurable set in R^n and k a nonnegative integer. A real measurable function u defined on D is said to have the *Lusin property of order* k if for any $\epsilon > 0$ there is a C^k-function g on R^n such that $|\{x \in D : u(x) \neq g(x)\}| < \epsilon$, where we use the notation $|A|$ to denote the Lebesgue measure of a set A in R^n. Unless explicitly stated otherwise a function defined on a measurable subset D of R^n will be assumed to be real measurable and finite almost everywhere on D. A classical theorem of Lusin states that measurable functions which are finite almost everywhere has the Lusin property of order zero, while Whitney shows in [15] that functions which are totally differentiable almost everywhere have the Lusin property of order 1.

We now describe characterizations given in [9] of functions which have Lusin property of order k. A function u defined on D is said to have an *approximate* $(k-1)$-*Taylor polynomial* at x if there is a polynomial $p(x;y)$ centered at x and of degree at most $k-1$ such that

$$\mathop{aplimsup}_{y \to x} \frac{|u(y) - p(x;y)|}{|y-x|^k} < +\infty;$$

while u will be said to be *approximately differentiable of order* k at x if there is a polynomial $p(x;y)$ centered at x and of degree at most k such that

$$\mathop{aplim}_{y \to x} \frac{|u(y) - p(x;y)|}{|y-x|^k} = 0.$$

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It is shown in [9] that each of the following two statements is equivalent to the statement that u has the Lusin property of order k on D:

(1) u has an approximate $(k-1)$-Taylor polynomial at almost every point of D;

(2) u is approximately differentiable of order k at almost every point of D.

For a nonnegative integer k and a real number $p \geq 1$, a function u defined on an open subset D of R^n is said to have the *strong (k,p)-Lusin property* on D if there is a postive constant C such that for any $\epsilon > 0$ there is a C^k-function g defined on D with $\|g\|_{k,p;D} \leq C$ such that if we let $E = \{x \in D : u(x) \neq g(x)\}$ then $|E| < \epsilon$ and $\|g\|_{k,p;E} < \epsilon$, where for a measurable subset S of D

$$\|g\|_{k,p;S} := \sum_{|\alpha| \leq k} \|D^\alpha g\|_{L^p(S)},$$

We refer to [16, p.2] for the standard notations concerning the multi-index α which appears in the preceding formula. It is clear that if a function u has the *strong (k,p)-Lusin property* on D then $u \in W_p^k(D)$. On the other hand, we have shown in [8] that if D is a Lipschitz domain, then functions of the Sobolev space $W_p^k(D)$ have the strong (k,p)-Lusin property.

We remark here that the strong $(1,1)$-Lusin property for $u \in W_1^k(D)$ is a consequence of a more general result of Michael [12] in connection with the theory of non-parametric surface area: Let u be a function of bounded variation with compact support on R^n, then for each $\epsilon > 0$, there is a Lipschitz function g on R^n such that $|\{x \in D : u(x) \neq g(x)\}| < \epsilon$ and $|Var(u) - Var(g)| < \epsilon$, where $Var(f)$ denotes the total variation of a function f.

We now turn to some recent ramifications of the *strong (k,p)-Lusin property*. For a function u defined on an open set D the maximal function of u, Mu, is defined by

$$Mu(x) := \operatorname*{Sup}_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap D} |u(y)|dy, \ x \in R^n,$$

where $B(x,r)$ is the ball with center x and radius r. For properties of maximal functions we refer to [14] and [16]. We introduce also a modified maximal function of u, M_1u, which is defined by

$$M_1u(x) := \operatorname*{Sup}_{o<r\leq 1} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap D} |u(y)|dy, \ x \in D.$$

If u is integrable on every bounded measurable subset of D, then, for $r > 0$, $M_1u(x) \leq M_0v(x)$ for $x \in B(0,r)\cap D$ with v being the function which coincides with u on $B(0,r+1)\cap D$ and vanishes outside. Since M_0v is finite almost everywhere on R^n, M_1u is finite almost everywhere on $B(0,r)\cap D$. Thus M_1u is finite almost everywhere on D. The Sobolev space

$W_p^k(D)$ will always be understood with D an open subset of R^n. We shall denote by $W_b^k(D)$ the space of all those functions which are integrable together with all their generalized partial derivatives up to order k on every bounded measurable subset of D. For $u \in W_b^k(D)$, the generalized partial derivatives $D^\alpha u, |\alpha| \leq k$, will sometimes be written as u_α. If $u \in W_b^k(D)$, then for almost all $x \in D, u_\alpha(x)$ is defined for all α with $|\alpha| \leq k$. For a real function u defined on D and $\lambda \geq 0, t \geq 0$ let

$$\begin{aligned} \mu(u;\lambda) &:= |\{x \in R^n : |u(x)| > \lambda\}|; \\ u^*(t) &:= Sup\{\lambda : \mu(\lambda) > t\}. \end{aligned}$$

The function u^* is called the non-increasing rearrangement of u. It is well known that (see, for example, [16, p.26]):

$$\mu(u; u^*(t)) \leq t. \tag{1}$$

Now we assume that there is $L > 0$ such that $|B(x,r)| \leq L|B(x,r) \cap D|$ for any $x \in D$ and $0 < r \leq 1$, that is, D is of type A in the sense of Campanato[2], although we do not assume D to be bounded. We show in effect the following Lusin type theorem in [11]:

Theorem 1. *There is a positive constant $C = C(n,k,L)$ such that for $u \in W_b^k(D)$ and $t > 0$, there exist $u_t \in C^k(R^n)$ and closed subset F_t of D so that*

i) $|D \setminus F_t| \leq 2t$;

ii) $u_\alpha(x) = D^\alpha u_t(x)$ for $x \in F_t, |\alpha| \leq k$; and

iii) $\|u_t\|_{W_\infty^k} \leq C(\sum_{|\alpha|\leq k} M_1 u_\alpha)^*(t)$.

As is shown in [11], it follows from Theorem 1 that the Sobolev space $W_p^k(D)$, $1 < p < +\infty$, is an interpolation space between the Sobolev spaces $W_1^k(D)$ and $W_\infty^k(D)$. This result is first given in [3] under more restrictive condition on D. We also indicate in [11] that the strong (k,p)-Lusin property of functions in $W_p^k(D)$ is a consequence of Theorem 1. We remark here that from the proof of the strong (k,p)-Lusin property of functions in $W_p^k(D)$ by using Theorem 1, the C^k-function g in the definition of the strong (k,p)-Lusin property is defined actually on R^n and hence this implies that $C^k(\overline{D})$ is dense in $W_p^k(D)$ in the case that D is a domain of type A. Hence Theorem 1 is an useful form of Lusin property and it is desirable to establish similar results for other function spaces. For an arbitrary open subset D of R^n we consider the space $L_0(D)$ of functions u such that

$$\lim_{\lambda\to\infty} |\{x \in D : |u(x)| \geq \lambda\}| = 0$$

and its subspaces $L^p_w(D), p > 0$, which consists of all those functions u for which there is a constant $C \geq 0$ such that

$$|\{x \in D : |u(x)| \geq \lambda\}| \leq C\lambda^{-p}.$$

For functions $u \in L^p_w(D)$ we denote by $N_p(u)$ the nonnegative number such that $N_p(u)^p$ is the smallest number C in the preceding definition. It is easy to see that $L_0(D)$ consists exactly of those functions u for which $u^*(t) < \infty$ for $t > 0$ and that

$$u^*(t) \leq N_p(u)t^{-1/p} \tag{2}$$

for $u \in L^p_w(D)$, hence $u^* \in L^p_w(R_+)$ and $N_p(u^*) \leq N_p(u)$ for $u \in L^p_w(D)$. Corresponding to Theorem 1 is the following theorem for $L_0(D)$:

Theorem 2. *For $u \in L_0(D)$ and $t > 0$ there exist closed subset F_t of D and continuous function u_t defined on R^n such that*

i) $|D \setminus F_t| \leq 2t$;

ii) $u(x) = u_t(x)$ *for* $x \in F_t$; *and*

iii) $\|u_t\|_{L^\infty} \leq u^*(t)$.

Since the proof for Theorem 2 is a simplified version of the proof for Theorem 3 in the following, we omit its proof. From Theorem 2 and (2) we have

Corollary 1. *In order for a function u defined on D to be in $L^p_w(D)$ it is necessary and sufficient that there is a constant $C > 0$ such that for each $t > 0$, there is a continuous function g defined on R^n with $\|g\|_{L^\infty} \leq t$ so that $|\{x \in D : u(x) \neq g(x)\}| \leq Ct^{-p}$.*

Using Theorem 2 we can give an interesting proof of the following corollary which does not seem to have been stated explicitly:

Corollary 2. *Let $u \in L^p(D), p \geq 1$ and let $\epsilon > 0$. Then there is a continuous function g defined on R^n such that $|\{x \in D : u(x) \neq g(x)\}| \leq \epsilon$ and $\|u - g\|_{L^p(D)} \leq \epsilon$.*

Proof. For $t > 0$ choose u_t and F_t as in Theorem 2. Then

$$\|u - u_t\|_{L^p(D)} \leq \|u\|_{L^p(D \setminus F_t)} + \|u_t\|_{L^p(D \setminus F_t)};$$

but we have from Theorem 2

$$\|u_t\|_{L^p(D\setminus F_t)} \leq [2tu^*(t)^p]^{1/p} = [2t(|u|^p)^*(t)]^{1/p} \leq [2\int_0^t (|u|^p)^*(s)ds]^{1/p},$$

hence, since $\int_0^\infty (|u|^p)^*(s)ds = \|u\|^p_{L^p(D)} < \infty$, we complete the proof by choosing g to be u_t for a sufficiently small t.

We introduce in [10] the spaces $Q^p_w, p \geq 1$ of functions defined on R^n and study their Lusin-type properties. Some of the results in [10] will be extended to more general spaces in the light of Theorem 1. We still denote by D an open set in R^n. For a function u defined on D and $x \in D$, let

$$q(u;x) = \underset{r>0}{Sup} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap D} |u(y) - u(x)|dy;$$

q(u;x) is called the *maximal mean steepness* of u at x. As we have argued in [10] for the case $D = R^n$, $q(u;\cdot)$ is measurable. For $0 \leq p < \infty$ denote by $Q^p_w(D)$ the space of functions u defined on D such that $q(u;\cdot) \in L^p_w(D)$, where we understand by $L^0_w(D)$ the space $L_0(D)$ when $p = 0$. For $u \in Q^p_w(D)$ define a function α_u by

$$\alpha_u(x) = |u(x)| + q(u;x), \ x \in D,$$

and for convenience, $L^p_w(D) \cap Q^p_w(D)$ will be denoted by $LQ^p_w(D)$; while $Q^p_w(R^n)$ and $LQ^p_w(R^n)$ will be denoted by Q^p_w and LQ^p_w respectively. In [10] Q^p_w are defined for $p \geq 1$, but this restriction on p is not necessary. In what follows we assume again that D is of type A in the sense of Campanato[2].

We now state and prove a theorem that complements Theorem 1 when $k = 1$:

Theorem 3. *There is a constant $C > 0$ depending only on n and L such that for $u \in LQ^0_w(D)$ and $t > 0$ there exist closed subset F_t of D and Lipschitz function u_t defined on R^n so that*

1. *$|D \setminus F_t| \leq 2t$;*
2. *$u_t(x) = u(x)$ for $x \in F_t$; and*

3. $\|u_t\|_{Lip} \leq C\alpha_u^*(t)$,

where

$$\|u_t\|_{Lip} = \|u_t\|_{L^\infty} + \underset{x \neq y}{Sup} \frac{|u_t(x) - u_t(y)|}{|x-y|}.$$

Proof. For $u \in LQ_w^0(D)$ and $t > 0$, let $W_t = \{x \in D : \alpha_u(x) \leq \alpha_u^*(t)\}$, then $|D \setminus W_t| \leq t$ by (1). For $x, y \in W_t$, by letting $r = |x - y|$, we have

$$|u(x) - u(y)| \leq 2\{|u(x)| + |u(y)|\}|x-y| \leq 4\alpha_u^*(t)|x-y|, \tag{3}$$

if $r \geq 1/2$; while if $r \leq 1/2$, we have

$$\begin{aligned}
|u(x) - u(y)| &= \frac{1}{|B(x,r) \cap D|} \int_{B(x,r)\cap D} |u(y) - u(x)|dz \\
&\leq L \frac{1}{|B(x,r)|} \int_{B(x,r)\cap D} |u(y) - u(x)|dz \\
&\leq L\left\{ rq(u;x) + 2^n \frac{1}{|B(y,2r)|} \int_{B(y,2r)\cap D} |u(z) - u(y)|dz \right\} \\
&\leq L[rq(u;x) + 2^{n+1} rq(u;y)] \leq 2^{n+2} L\alpha_u^*(t)|x-y|.
\end{aligned}$$

The last inequality and (3) show that if we choose a closed subset F_t of W_t such that $|D \setminus F_t| \leq 2t$, then we complete the proof by letting $C = 22^{n+2}L = 2^{n+3}L$, because then $\|u|_{F_t}\|_{Lip} \leq C\alpha_u^*(t)$ and $u|_{Ft}$ can be extended to be a Lipschitz function u_t defined on R^n such that $\|u_t\|_{Lip} = \|u|_{F_t}\|_{Lip}$.

It follows then from Theorem 3 and (2) the corollary:

Corollary 3. *There is a constant $C > 0$ depending only on n, L and $p > 0$ such that for $u \in LQ_w^p(D)$ and $\lambda > 0$ there exist closed subset F_λ of D and Lipschitz function u_λ defined on R^n so that*

1. $|D \setminus F_\lambda| \leq C[N_p(u)^p + N_p(q(u;\cdot))^p]\lambda^{-p}$;
2. $u_\lambda(x) = u(x)$ *for* $x \in F_\lambda$; *and*
3. $\|u_\lambda\|_{Lip} \leq \lambda$.

If a domain D is minimally smooth in the sense of Stein [14], then there is a constant $C > 0$ depending only on D such that every function $u \in W_1^1(D)$ can be extended to be a function $\overline{u}$ with $\|\overline{u}\|_{W_1^1} \leq C\|u\|_{W_1^1(D)}$; from this and the well-known fact that if $u \in L^1(D) \cap BV(D)$ then there is a sequence g_k in $C^1(D)$ such that $\lim\limits_{k\to\infty} \|u - g_k\|_{L^1(D)} = 0$ and $\operatorname*{Lim}\limits_{k\to\infty} \|Dg_k\|_{L^1(D)} = Var(u)$, it follows that if $u \in L^1(D) \cap BV(D)$, then $u \in LQ^1_w(D)$ and $N_1(q(u;\cdot)) \leq CVar(u)$ (see [10]). Thus we have

Corollary 4. *Let D be a minimally smooth domain. Then there is a constant $C > 0$ depending only on D such that for $u \in L^1(D) \cap BV(D)$ and $\lambda > 0$ there exist closed subset F_λ of D and Lipschitz function u_λ defined on R^n so that*

1. $|D \setminus F_\lambda| \leq C\|u\|_{BV(D)}\lambda^{-1}$;
2. $u_\lambda(x) = u(x)$ *for* $x \in F_\lambda$; *and*
3. $\|u_\lambda\|_{Lip} \leq \lambda$.

We point out in concluding our note that Lusin-type properties of functions have various kind of applications. For some of the applications we refer to [1], [4], [5], [6], [7], [9], and [11].

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