SOME SLIGHTLY SUBCRITICAL OR SLIGHTLY SUPERCRITICAL PROBLEMS

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The main purpose of the present article is to discuss the existence of positive solutions of the exterior problem

$$\begin{aligned} -\Delta u &= u^{p^*-\epsilon} \text{ in } R^m \backslash \Omega \\ u &= 0 \text{ on } \partial\Omega \\ u &> 0 \text{ on } R^m \backslash \Omega \end{aligned} \tag{1}$$

where $p^* = (m+2)(m-2)^{-1}$ and $m \geq 3$.

Our main result is the following.

Theorem 1. *Assume that $m = 3, 4$ or $6, \Omega$ is a bounded open set in R^m with smooth boundary such that $R^m \backslash \Omega$ is connected and the reduced homology $\widetilde{H}_*(\Omega, Z_2)$ is non-trivial. Then for ϵ small and positive, (1) has a solution*

Remarks 1. The condition on m is only needed for technical reasons and should not be necessary. It is only needed to ensure that the solutions of (1) with $\epsilon = 0$ have a good local structure. In particular, this holds if the solutions of (1) with $\epsilon = 0$ are isolated (for an appropriate norm).

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2. Note that solutions of (1) necessarily decay at infinity by Lemma 2 in [6].

3. The reduced homology conditions always holds if Ω is not connected. In [4], it is shown that there exists Ω satisfying the assumptions of Theorem 1 but (1) only has a solution for ϵ small. We do not know if there is such an Ω with Ω connected. If Ω is star shaped, (1) never has a solution for $1 < p^* - \epsilon < p^*$. Thus some condition on Ω is necessary. There are examples of contractible Ω for which (1) has a solution. In general it appears complicated to decide for which Ω and ϵ (1) has a solution.

4. The problem arose in [4] as a limiting problem for problems on bounded domains with small holes.

It turns out that our techniques can also be used to establish the following theorem.

Theorem 2. *Assume that $m = 3, 4$ or 6 and Ω is a smooth bounded domain such that the reduced homology $\widetilde{H}_*(\Omega, Z_2)$ is non-trivial. Then the problem*

$$\begin{aligned} -\Delta u &= u^{p_*+\epsilon} \quad \text{in } \Omega \\ u &= 0 \quad\quad \text{on } \Omega \\ u &> 0 \quad\quad \text{in } \Omega \end{aligned} \tag{2_ϵ}$$

has a solution for small positive ϵ.

Remark. Once again the condition on m is probably unnecessary. We do not have any example showing that ϵ must be small though, if $m > 3$, Passaseo [7] has an example showing that the result need not be true for every $\epsilon > 0$. Indeed it is expected that the result is probably true for $\epsilon \leq t_m$ where t_m depends only on m.

The idea of the proof of Theorem 1 is simple though the details are complicated. The Bahri-Coron theorem implies that (2_0) has a positive solution. We essentially prove that (2_0) has a solutions which persists under small perturbations. Now solutions of (2_0) are smooth on $\overline{\Omega}$ and solutions which are close in $\dot{W}^{1,2}(\Omega)$ are uniformly close (cp part of the proof of Theorem 3 in [5]). On $C(\Omega)$, the right hand side defines a real analytic mapping. This is where we use our assumptions on m. Thus, on $C(\Omega)$, the solutions of (2_0) are the zeros of a real analytic compact

Fredholm map and hence by [3] the solutions T of (2_0) have the following nice properties. Components of T are path connected (in fact points in a component can be connected by piecewise smooth continuous curves). By differentiating the natural energy of (2_0) along such a curve, following that the energy is constant on a component. Moreover, since there are only a finite number of components intersecting each ball in $C(\Omega)$, E takes only countably many values on its critical points (i.e. on solutions of (2_0). This is one minor point in the above argument. Zero is an isolated solution of (2_0) and it is easy to see that the set of positive solutions of (2_0) is an open and closed set in the set of all solutions, Thus the positivity condition causes no problem.

Without loss of generality, we can assume $0 \in \Omega$. By an inversion about 0 (cp [4]), (3) is equivalent to the problem

$$\begin{aligned} -\Delta v &= \|y\|^{-q(p^*-\epsilon)} v^{p^*-\epsilon} && \text{in } \Omega^*\backslash\{0\} \\ v &= 0 && \text{on } \partial\Omega^* \\ v &> 0 && \text{in } \Omega^*\backslash\{0\} \end{aligned} \qquad (3_\epsilon)$$

where $q(p) = (m+2) - p(m-2)$ and $\Omega^* = \{\|x\|^{-2}x : x \in R^m\backslash\Omega\} \cup \{0\}$. As shown in [4], (1) is equivalent to (3_ϵ) and the condition on the homology of Ω in Theorem 1 is equivalent to the reduced homology of Ω^* being non-trivial. We will construct solution of (3_ϵ) which are positive and continuous at zero. Note that if p is near p^*, we can think of $\|y\|^{-q(p)}v^p$ as a small perturbation of v^{p^*} and we expect solutions of (3_ϵ) near those of (2_0) (for Ω replaced by Ω^* in (2_0). Now it is shown in [5] that an isolated solution of (2_0) (with Ω replaced by Ω_*) with non-trivial Morse numbers (or indeed suitable compact sets of solutions) continues to solutions of (3_ϵ). This is provided by using a truncated version of (3_ϵ). Since the Bahri-Coron theorem is proved by relative homology calculations, it is natural to expect that a solution of (2_0) (for Ω replaced by Ω^*) with the above properties to exist and hence we expect that (3_ϵ) has a solution. If every component of the solutions of (2_0) (with Ω replaced by Ω^*) is compact, and if no component satisfies the non-trivial Morse number condition, we prove that the calculations for the relative homologies

$H_*(E_b, E_a)$ are unaffected and hence we can obtain a contradiction much as in Bahri [1]. Here E is the natural energy for (2_0) (with Ω replaced by Ω^*) and $E_b = \{u : \dot{W}^{1,2}(\Omega^*) : E(u) \leq b\}$. The proof calculates $H_*(E_b, E_a)$ and eventually obtains a contradiction by choosing b large and a close to zero. Note that both Bahri [1] and Bahri-Coron [2] work with a reduced energy on the unit sphere in $\dot{W}^{1,2}(\Omega^*)$ but it is easy to change from one formulation to the other.

There are two major complications in the above argument which need to be considered. Firstly, the presence of these topological trivial solutions u_0 produce extra levels where the Palais-Smale condition may fail (at the levels $E(u_0) + km^{-1}S^{\frac{1}{2}m}$ for k a positive integer). Here S is defined in Bahri-Coron [2]. This leads to the possibility that the Palais-Smale condition may fail at a countable number of points with only limit points in $\{km^{-1}S^{\frac{1}{2}m}\}$. However, in this case we use ideas in [2] to show that crossing one of the above levels does not change relative homologies. (Here for simplicity, I am assuming $E(u_0)$ is not an integer multiple of $m^{-1}S^{\frac{1}{2}m}$). Essentially one proves that near such levels and near non Palais-Smale sequences our mappings can be nearly split as a product.

The second complication is that at levels where the Palais-Smale conditions fail, there may be non-compact components of the solutions of (2_0) (for Ω replaced by Ω^*). To make our argument work, we need to consider two cases for such a component T. Firstly for any compact subset K of T and for all ϵ small there is a solution near K of a truncated version of (3_ϵ) and hence of (3_ϵ) (as in [4]). In this case, we are finished. If not, we show by piecing together deformations (including one determined by the truncated version of (3_ϵ)) that we can deform down across the energy level C (where $E(T) = C = km^{-1}S^{\frac{1}{2}m}$) except near "$\infty$" (and in fact except in $V(P, \epsilon)$ with the notation of [1]) where we can then use the Bahri ideas from [1] to calculate the change in relative homology.

There are extra complications in that the two difficulties may interact at the same energy level, the non Palais-Smale sequences at the same energy level may be generated by solutions at different lower energy levels and it is necessary to check carefully that some small perturbations we have to make do not affect the Bahri argument. Thus the full details are longer and rather tedious. They will appear elsewhere.

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